half of Thm 4: $G$ is connected $\Rightarrow$ $G$ has a spanning tree

Several proofs that construct the tree by cutting a edge at a time from $G$.

LEMMA If $G$ is connected, $E$ an edge in some circuit in $G$ and $H$ is $G$ with $E$ deleted, then $H$ is connected.

Proof (4): Let $G$ be connected. $G$ has alot of connected subgraphs with all the vertices of $G$. Let $H$ be a connected subgraph of $G$ with all of $G$'s vertices with as ~~fewer~~ few edges as possible. If $H$ is a tree, then it's a spanning tree.

If $H$ isn't a tree, by thm 2, $H$ has a circuit. Let $E$ be an edge in a circuit of $H$. Let $H'$ be $H$ with $E$ deleted. the Lemma says $H'$ is connected and has 1 less edge than $H$. But this contradicts the fact that $H$ has as few edges as possible. Therefore $H$ is a spanning tree of $G$.

Proof (5): Let $G$ be connected. Let $G$ have $n$ vertices and $e$ edges and let $m = e-n+1$. For $i = 0, 1, \dots m$, we will construct a connected subgraph $H_i$ of $G$ that has all of $G$'s vertices and $e-i$ edges. For ~~$H_0$~~ $i=0$, $H_0 = G$. Now assume $H_i$ ~~is~~ has been constructed, for $i<m$. $H_i$ has $e-i > e-m = n-1$ edges, so it isn't a tree. But $H_i$ is connected, so it must have a circuit. Delete an edge in this circuit to obtain $H_{i+1}$.

Now $H_m$ is connected, has $n$ vertices and $n-1$ edges. It can be shown that this implies $H_m$ is a tree. Also we need to know from the start that $e \geq n-1$, and this needs proof (induction proof is easy (on the number of vertices))

Proof (6): By induction on the number of ~~vertices~~ $k$ circuits of $G$. If $k=0$, $G$ is a tree and hence its own spanning tree. Assume it is true for all connected graph with $\leq k$ circuits. Let $G$ have $k+1$ circuits. Use the lemma to construct $H$ by deleting an edge from some circuit in $G$. Having fewer circuits $H$ & hence $G$ has a spanning tree by the inductive hypothesis.

Proof (3): By induction on the number of vertices.
Start up: $G$ has 1 vertex, then $G$ is a tree and hence its own spanning tree.

Induction step: Assume each connected graph with $n$ or fewer vertices has a spanning tree. Let $G$ be a connected graph with $n+1$ vertices. Let $x$ be any vertex of $G$ and let $H$ be the subgraph obtain from $G$ by deleting $x$ (and the edges incident at $x$)

In general, $H$ need not be connected. ($x$ could be an articulation point). So let $H_1, H_2, \ldots H_k$ be the components of $H$. Each of the components $H_i$ has $n$ or fewer vertices. Hence, by inductive hypothesis $H_i$ has a spanning tree $T_i$. Next we show how to glue these into a spanning tree for $G$.

Note that the edges in $G$ that are not in $H$ go from $x$ to one of the $H_i$'s. Further, if $z$ is a vertex in $H_i$, then there is a path $x \to z$ (since $G$ is connected) and hence a edge $x \mapsto$ some vertex in $H_i$. Thus $T$ is constructed by using $x$, each of the $T_i$'s and exactly one edge $x \to$ some vertex in $H_i$ for each $i$.

$T$ clearly contains all the vertices of $G$. $T$ is connected, since each $T_i$ is connected and there is a path joining pts in $T_i$ to $T_j$ through $x$ when $i \neq j$. Also $T$ has no circuits, since each $T_i$ is a tree and a path that leaves a $T_i$ can never come back (there is one way to $x$). Therefore $T$ is a spanning tree

<u>half of Thm 4</u>: $G$ is connected $\Rightarrow$ $G$ has a spanning tree

Several proofs that <u>build</u> the tree <u>up</u> by adding a edge at a time.

<u>LEMMA</u>: If $G$ is connected, $T$ is a subgraph which is a tree but not a spanning tree, then there is an edge $E$ in $G$ but not $T$, so that $T$ with $E$ added is also a tree.

<u>Proof (1)</u>: Let $G$ be connected. $G$ has alot of subgraphs which are trees. Let $T$ be a tree subgraph of $G$ with as many edges as any other tree subgraph of $G$. (i.e $T$ is a maximal tree in $G$.)

Claim that $T$ is a spanning tree of $G$. For if this wasn't the case, we could use the lemma to construct $T'$ a tree with one more edge than $T$. But this contradicts the fact that $T$ has as many edges as any tree subgraph of $G$. Therefore $T$ is a spanning tree of $G$.

<u>Proof (2)</u>: Let $G$ be connected. Let $n+1$ be the number of vertices in $G$. For $0 \le i \le n$ we will construct (by induction) trees $T_0, T_1 \dots T_n$, so that each is a subgraph of $G$ and $T_i$ has exactly $i$ edges and $i+1$ vertices

Let $T_0$ be any one vertex from $G$ (this is the start up). (Now for the induction step). ~~If~~ assume $i < n$, and $T_i$ has been constructed. Now $T_i$ has $i+1 < n+1$ vertices, so it isn't a spanning tree. Thus the lemma gives us an edge $E$ to add to $T_i$ to make $T_{i+1}$ with $i+1$ edges and $i+2$ vertices.

Since $T_n$ has $n+1$ vertices, it's a spanning tree for $G$.