Definition 1: A graph is a tree $\Longleftrightarrow$ $\forall x,y$ vertices with $x \neq y$ then there is a unique path from x to y.

Definition 2: A graph is connected $\Longleftrightarrow$ $\forall x,y$ vertices with $x \neq y$ then there is a path from x to y.

Thm 1: A tree has no circuits

Thm 2: A connected graph with no circuits is a tree

Thm 3: The ~~removable~~ of any edge disconnects a tree

Definition 3: a tree T which is a subgraph (subset of edges) of a graph G is a spanning tree if all the vertices of G are in T.

Thm 4: G is connected if and only if G has a spanning tree

Thm 5: A tree is planar

Thm 6: Euler's formula shows a tree has one more vertex than edges.

Thm 7: a connected graph with n vertices has at least n-1 edges.

Thm 7½: If the removal of any edge disconnects a graph, then its a tree

Thm 8: a connected graph with fewer edges than vertices is a tree

Thm 9 The removal of any edge of a tree divides the graph into two components.

Thm 10: Any tree with more than one edge has at least two vertices with degree one

Thm 11: Any connected graph has a vertex which is not an articulation point.

Thm 12: A graph with n-vertices, n-1 edges and no circuits is a tree

Rules: In proving Thm i you can use only those results above it. Or any result in the text before Chapter 9.

[illegible] the [illegible]

[illegible] graphs [illegible] not connected [illegible] G, then E is a [illegible]

[illegible] A cut edge E [illegible] graph G is [illegible] spanning tree of G

[illegible] A graph which [illegible] has no cut edges

Thm E: The converse [illegible] true.

Thm F: The statement [illegible] articulation point of the connected graph [illegible] in no circuits in G" is [illegible]

Thm G: The statement "If [illegible] vertex of a connected graph and x is in no [illegible], then x is an articulation [illegible]" isn't true.

Thm H: A vertex x of [illegible] articulation point if and only if deg x > 1.

Thm I: A vertex x of [illegible] graph G is an articulation point if and only if [illegible] vertices a and b in G such that every path joining a and b passes through x.

[illegible] An edge E of [illegible] graph G is a cut edge if and only if there [illegible] in G such that every path joining a and [illegible]

[illegible] connection any tree [illegible]

[illegible] without crossing [illegible]

[illegible] graph G so that [illegible] exactly one edge [illegible] subset [illegible] of G

A. ~~Definition: A path P in a Graph is a sequence of vertices $x_1, x_2, \ldots x_n$ s.t. x_i is ~~connected~~ adjacent to x_{i+1} for $1 \le i \le n-1$ and no edge is used more than once~~

A Definition: a tour T is a sequence of one ~~or~~ or more vertices s.t. x_i is adjacent to x_{i+1} for all i

B Definition a path P is a tour ~~&~~ s.t. no edge is repeated (in either direction)

C. Definition a simple path S is a path that doesn't repeat vertices.

D Definition a circuit C is a path that starts and stops at the same vertex AND uses at least one edge!

E. Definition a simple circuit is a circuit in which only the starting vertex is repeated.

1 Disprove: If P, Q are different paths: $x \to y$, then C given by $P: x \to y$ followed by Q backwards $y \to x$ is a circuit.

2 Lemma: If P, Q are different path $x \to y$ in G, then G has a circuit.

3. Disprove: If T is a tour that starts and stop at x in G then there is a circuit C in G which stops & starts at x.

4. Lemma: If E is a edge from x to y and P is a path $x \to y$ which does not use E, then P followed by E is a circuit.

5 ~~Lemma~~ Disprove: If P is a path $x \to y$ and Q is a path $y \to z$, then P followed by Q is a path $x \to z$.

6 __Lemma__: If P is a path $x \to y$ in G and Q is a path $y \to z$ in G then there is a path $x \to z$ in G

F __Definition__: A tree T is a graph so that for each pair of vertices $x, y \in T$ there is a unique path $x \to y$ in T (note $x = y$ is allowed) [Compare w/ directed & ordered trees]

7. __Theorem__: A tree has no circuits

8. __Theorem__: Any path $P : x \to y$ in a tree is a simple path

9. __Disprove__: A graph with no circuits is a tree.

G. __Definition__: A graph G is connected if for each pair of vertices $x, y \in G$ there is a path $x \to y$ in G.

10. __Lemma__: A tree is connected

11 __Lemma__: A connected graph G with no circuits is a tree.

12 __Theorem__: A graph T is a tree if and only if it is connected and has no circuits

H. __Definition__ If G is a connected graph and the edge E in G is so that if E is removed then G is disconnected then E is called a __cut edge__.

13. __Disprove__ If E is a cut-edge of G and goes $x \to y$, then E is the unique path $x \to y$ in G.

14. __Lemma__: An edge in no circuit is a cut-edge (in a connected graph G) ~~A cut-edge is in no circuit~~

15. __Lemma__: A cut-edge is in no circuit

16. __Theorem__: A graph T is a tree if and only if it is connected and every edge is a cut edge.

17 Lemma: If E is an edge $x \to y$ in a connected graph G, then E is a cut-edge if and only if every path $x \to y$ uses E

18. Lemma If E is an edge $x \to y$ in a connected graph G, then E is a cut-edge if and only if there are vertices w, z in G s.t. every path $w \to z$ uses E.

I Definition: A directed graph G is said to be a directed tree if there is a vertex called the root and ~~ther~~ for each vertex $x \in G$ there is a unique path which follows the direction of the arrows from the root to x. ($x = \text{root}$ is possibility)

19 Lemma: In ~~an~~ directed tree, the in-degree of the root is zero.

20 Lemma: In a directed tree, the in-degree of any vertex other than the root is one.

21 Lemma: If in the directed tree T we forget the directions of the edges and that one vertex was special, we have a tree in the usual sense.

22. Lemma: If T is a tree, and we pick one vertex of T to be the root and direct all the edges away from this vertex we have a directed tree.

23. Lemma: A directed tree has one more vertex than edges

24. Theorem: A tree with n vertices has $n-1$ edges

J Definition: A subgraph T of a graph G is a spanning tree for G if T is a tree and contains all the vertices of G

25. <u>Theorem</u> A graph with a spanning tree is connected

26. <u>Lemma</u>: If G is a connected, and T is a subgraph of G which is a tree but not a spanning tree of G then there is some edge E of G which is not in T so that if T' is T with E adjoined then T' is a tree.

27. <u>Lemma</u> If G is a connected graph and H is a connected subgraph of G which is not a tree then there is some edge E of H so that if H' is H with E removed ~~is still~~ then H' is still connected

28. <u>Theorem</u>: ~~If~~ G is connected if and only if G has a spanning tree.

29. <u>Corollary</u>: If G is connected and has n vertices, then G has at least $n-1$ edges

30 <u>Theorem</u>: A connected graph with n vertices and $n-1$ edges is a tree.

31. <u>Theorem</u>: A Graph is a tree if and only if it is connected and has fewer edges than vertices.

32. <u>Lemma</u>: A maximal tree subgraph of a connected graph is a spanning tree

33. <u>Lemma</u>: A minimal connected subgraph containing all the vertices of a connected graph is a spanning tree.

34.K <u>Definition</u>: A maximal connected subgraph of G is called a component.

44 Theorem: ~~If~~ The vertex x in a tree T is an articulation point if and only if $\text{degree}(x) \geq 2$

45 Theorem: Each connected graph has a non-articulation point.

46, Proposition: If G is connected and x is a vertex of G, Then H (G with x & incident edges removed) consists of a certain number of components say $H_1, \dots, H_n$ ($n=1$ is possible). Each of the removed edges from G go from x to one of the H_i's and there is at least one edge to each H_i.

47. Corollary: If G is connected and x is a vertex of degree n then H has at most n components where H is G with x and incident edges removed.

48 Disprove: A connected graph with a cut-edge has an articulation point

49 Lemma: A connected graph with at least three vertices and a cut-edge has an articulation point

50, Disprove: A connected graph with an articulation point has ~~an~~ a cut-edge.

half of Thm 4: G is connected $\Rightarrow$ G has a spanning tree

Several proofs that build the tree up by adding a edge at a time.

LEMMA: If G is connected, T is a subgraph which is a tree but not a spanning tree, then there is an edge E in G but not T, so that T with E added is also a tree.

Proof (1): Let G be connected. G has alot of subgraphs which are trees. Let T be a tree subgraph of G with as many edges as any other tree subgraph of G. (i.e T is a maximal tree in G.)

Claim that T is a spanning tree of G. For if this wasn't the case, we could use the lemma to construct T' a tree with one more edge than T. But this contradicts the fact that T has as many edges as any tree subgraph of G. Therefore T is a spanning tree of G.

Proof (2): Let G be connected. Let $n+1$ be the number of vertices in G. For $0 \le i \le n$ we will construct (by induction) trees $T_0, T_1, \dots T_n$, so that each is a subgraph of G and T_i has exactly i edges and $i+1$ vertices

Let T_0 be any one vertex from G (this is the start up). (Now for the induction step). Assume $i < n$, and T_i has been constructed. Now T_i has $i+1 < n+1$ vertices, so it isn't a spanning tree. Thus the lemma gives us an edge E to add to T_i to make T_{i+1} with $i+1$ edges and $i+2$ vertices.

Since T_n has $n+1$ vertices, it's a spanning tree for G.

Note that the edges in G that are not in H_i

go from x to one of the H_i's. Further, if z is a vertex in H_i, then there is a path $x \to z$ (since G is connected) and hence a edge $x \mapsto$ some vertex in H_i

Thus T is constructed by using x, each of the T_i's and exactly one edge $x \to$ some vertex in H_i for each i

half of Thm 4: G is connected $\Rightarrow$ G has a spanning tree

Several proofs that construct the tree by cutting a edge at a time from G.

LEMMA If G is connected, E an edge in some circuit in G and H is G with E deleted, then H is connected.

Proof (4): Let G be connected. G has alot of connected subgraphs with all the vertices of G. Let H be a connected subgraph of G with all of G's vertices with as ~~fewer~~ few edges as possible. If H is a tree, then it's a spanning tree.

If H isn't a tree, by thm 2, H has a circuit. Let E be an edge in a circuit of H. Let H' be H with E deleted. the Lemma says H' is connected and has 1 less edge than H. But this contradicts the fact that H has as few edges as possible. Therefore H is a spanning tree of G.

Proof (5): Let G be connected. Let G have n vertices and e edges and let $m = e - n + 1$. For $i = 0, 1, \dots m$, we will construct a connected subgraph H_i of G that has all of G's vertices and $e - i$ edges. For $i = 0$, $H_0 = G$. Now assume H_i has been constructed, for $i < m$. H_i has $e - i > e - m = n - 1$ edges, so it isn't a tree. But H_i is connected, so it must have a circuit. Delete an edge in this circuit to obtain H_{i+1}.

Now H_m is connected, has n vertices and $n-1$ edges. It can be shown that this implies H_m is a tree. Also we need to know from the start that $e \geq n-1$, and this needs proof (induction proof is easy (on the number of vertices))

Proof (6): By induction on the number k of ~~vertices~~ circuits of G. If $k = 0$, G is a tree and hence its own spanning tree. Assume it is true for all connected graph with $\leq k$ circuits. Let G have $k+1$ circuits. Use the lemma to construct H by deleting an edge from some circuit in G. Having fewer circuits H & hence G has a spanning tree by the inductive hypothesis.