~~DM2~~ MAD 3105 -01 PROF BELLENOT
213 LOVE
MWF 2:30 - 3:30

(The good doctor is available most afternoons (if he is in))

TEXT: MOTT, KANDEL, BAKER "Discrete Math for ..."
(Is it the 2nd Edition?) Chapts 3,4,7 & rest of 2 k5

IMPORTANT DATES:

FINAL @ 3:00pm MON 8 DEC.

MIDTERMS 1 OCT & 5 NOV (TENTATIVE)

GRADES: 90, 80, 70, 60 percentage cut offs based on FINAL 36% MIDTERMS (18% EACH) TP's (18%) AND HW 10%.

HOMEWORK: HW is given each class to be turned in the next class. Late HW isn't accepted. Your HW grade is ~~based on~~ min (100%, HW done / .90 x HW assigned)

TP's: Homework which is carefully graded on a 0-10 ~~base~~ basis. Assigned a week in advance of its due date. (Only the top 2/3's count towards your TP score.)

YOU NEED A "C-" or better in MAD 3104 to take this course.

A. R is p.o so its REF, ANTI-SYM & TRANS

$x SR y \iff xRy$ & $x \neq y$

<u>SR is irreflexive</u>: since $x = x$ is always true, both $x \neq x$ and xRx & $x \neq x$ is always false. ∴ $x \cancel{SR} y$

<u>SR is asymmetric</u>: Suppose not. Then $xSRy$ and $ySRx$, hence xRy, $x \neq y$, yRx & $x \neq y$. But R is anti sym, so xRy & $yRx \Rightarrow x = y$, an contradiction.

<u>SR is transitive</u>: If $xSRy$ & $ySRz$ then xRy, $x \neq y$ yRz and $y \neq z$. If $x = z$ then xRy & yRx are true (by replacing z with x); but then $x = y$ by R's anti-sym which contradicts $x \neq y$. Thus $x \neq z$ and since R is trans $xRz \Rightarrow xSRz$.

R is irref, asym & trans

$xREy \iff xRy$ or $x = y$. Show RE is p.o.

<u>RE is reflexive</u>

since $x = x$ is always true $x = x$ or xRx is also always true ∴ $xREx$

<u>RE is anti-sym</u>

Suppose not then there are $x \neq y$ with both $xREy$ & $yREx$. Since $x \neq y$, $xREy \Rightarrow xRy$ and $yREx \Rightarrow yRx$ But this contradicts R's asymmetry.

<u>RE is trans</u>: Suppose $xREy$ & $yREz$. $xREy \iff xRy$ or $x = y$. If $x = y$, then we can replace y by x and get $xREz$. So assume $x \neq y$. If $y = z$, then we can replace y by z and get $xREz$. so assume $y \neq z$. Hence both xRy & yRz are true. But since R is trans xRz and $xREz$ are true.

12 Sept 86 Some Answers:

11

	upper bounds	lub	lower bounds	glb
c	NONE	NONE	1	1
d	NONE	NONE	1	1
e	error $10 \notin D_{12}$ 12	if $B=\{2,6\}$ 12	NONE	NONE
f	NONE	NONE	1	1

14. A. Reflexive: $x \le x$ & $y \le y$ hence $(x,y) \preceq (x,y)$. Thus $\preceq$ is reflexive

Anti-Symmetry: if $(x,y) \preceq (w,z)$ & $(w,z) \preceq (x,y)$ then $x \le w$, $y \le z$, $w \le x$ & $z \le y$. Thus $x=w$ & $y=z$. Hence $(x,y)=(w,z)$ and $\preceq$ is anti-sym.

Transitive: if $(x,y) \preceq (r,s)$ & $(r,s) \preceq (w,z)$ then $x \le r$, $y \le s$, $r \le w$, $s \le z$. Hence $x \le w$ & $y \le z$ and $(x,y) \preceq (w,z)$.

NOT TOTAL ORDER: NEITHER $(0,1) \preceq (1,0)$ nor $(1,0) \preceq (0,1)$ is true.

B. (*) $\text{lub}\big((x,y),(w,z)\big) = (\max\{x,w\}, \max\{y,z\})$

$\text{glb}\big((x,y),(w,z)\big) = (\min\{x,w\}, \min\{y,z\})$

To see that ~~this~~ (*) is true: Note that $(\max\{x,w\}, \max\{y,z\})$ is an upper bound to $\{(x,y),(w,z)\}$ since $x,w \le \max\{x,w\}$, $y,z \le \max\{y,z\}$. To see that it is the lub, let (r,s) be any other upper bound. Thus $(x,y) \preceq (r,s)$ & $(w,z) \preceq (r,s)$ and hence $x,w \le r$ and $y,z \le s$ or $\max\{x,w\} \le r$ & $\max\{y,z\} \le s$. Therefore $(\max\{w,x\}, \max\{y,z\}) \preceq (r,s)$ and it is lub.

The glb. is done similarly.

TP 8: Proof: There are two cases to consider either $x=y$ or $x \neq y$

if $x=y$ then since T & R are reflexive xTx & xRx

hence "$xTy \iff xRy$" is true in this case

else $x \neq y$ now $xTy \iff xSy$ or $x=y \iff xSy$

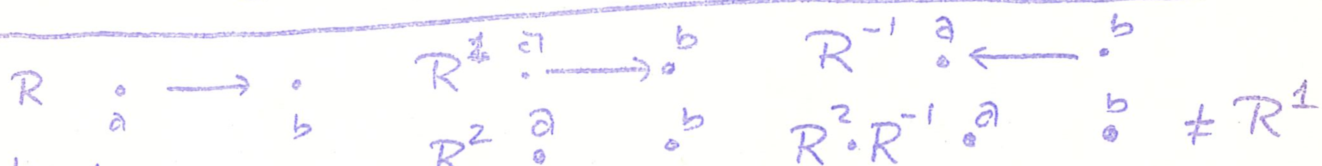
and $xRy \iff xRy$ and $x \neq y \iff xSy$

hence $xTy \iff xRy$ in this case too

$\therefore$ $xTy \iff xRy$. And $R=T$

EXAMPLE: (Shows that R must be a p.o. for TP8 to work, that is the use of R being reflexive is essential to the proof.)

12.



$k=1$

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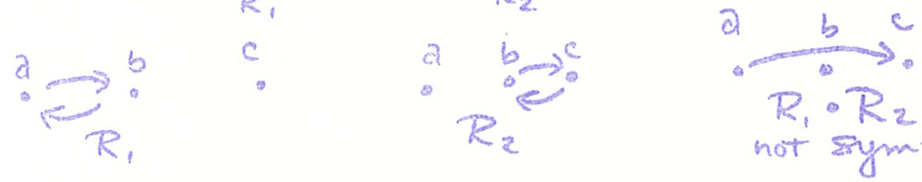
14 A. let $x \in A$. xR_1x & $xR_2x \Rightarrow xR_1 \cdot R_2x$ $\therefore$ $R_1 \circ R_2$ is ref.

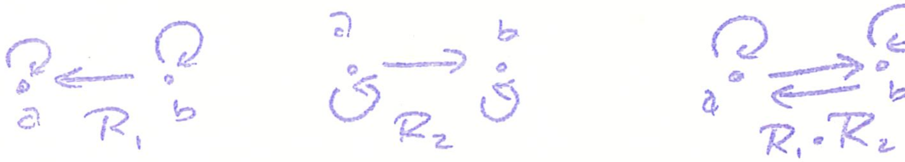
B. Counter example

C ~~Suppose~~



not symmetric

D.



not anti-symmetric

E.

not trans.

More on lattices. A poset $A \leq$ is a lattice if each pair of elements has a g.l.b. and l.u.b. Sometimes it is easier to write $\text{glb}\{x,y\} = z$ as $z = x \wedge y$ and $\text{lub}\{x,y\} = w$ as $w = x \vee y$.

Examples 1. $\mathcal{P}(X), \subseteq$ is a lattice. Think of x & y as ~~subspa~~ subets of X. $x \vee y$ is $x \cup y$ and $x \wedge y$ is $x \cap y$. $x \cap y$ is a lowerbdd to $\{x,y\}$ because $x \cap y \subseteq x$ & $x \cap y \subseteq y$. $x \cap y$ is the glb since in addition if $z \subseteq x$ & $z \subseteq y$ then $z \subseteq x \cap y$.

2. $\mathbb{N}, |$ is a lattice ($|$ is the relation "divides") Think of x, y as integers ≥ 1. $x \vee y$ is the ~~g.c.d.~~ l.c.m., the ~~greatest~~ least common multiple, the ~~biggest~~ smallest number z that ~~divides~~ both x & y divide into z. $x \wedge y$ is the g.c.d. the great common divisor. the largest number that divides both x & y.

3. (From Digital) the switching algebra $\{0,1\} \leq$ x & y are either 0 or 1 $x \vee y$ is written $x+y$ and $x \wedge y = xy$. [see digital for details]

4. Boolean Algebras [similar to 3 above]

5. $\mathbb{R} \leq$ {reals with the usual stuff} $x \wedge y = \min\{x,y\}$. $x \vee y = \max\{x,y\}$ (Totally Ordered sets are lattices.)

11. Prove by mathematical induction that if R is a relation on A, then
   (a) $R^m \cdot R^n = R^{m+n}$
   (b) $(R^m)^n = R^{mn}$
12. Give an example of a relation R and a positive k such that $R^{k+1} \cdot R^{-1} \neq R^k$.
13. (a) Show that the transitive closure of a symmetric relation is symmetric.
   (b) Is the transitive closure of an antisymmetric relation always antisymmetric?
   (c) Show that the transitive closure of a reflexive and symmetric relation is an equivalence relation.
14. Let R_1 and R_2 be arbitrary binary relations on a set A. Prove or disprove the following assertions.
   (a) If R_1 and R_2 are reflexive, then $R_1 \cdot R_2$ is reflexive.
   (b) If R_1 and R_2 are irreflexive, then $R_1 \cdot R_2$ is irreflexive.
   (c) If R_1 and R_2 are symmetric, then $R_1 \cdot R_2$ is symmetric.
   (d) If R_1 and R_2 are antisymmetric, then $R_1 \cdot R_2$ is antisymmetric.
   (e) If R_1 and R_2 are transitive, then $R_1 \cdot R_2$ is transitive.
15. Let R be a binary relation on a set A where A has n elements. Prove that the transitive closure of $R = \bigcup_{i=1}^{n} R^i$.
16. Prove that if R is a transitive relation on a set A, then for each positive integer n, $R^n \subseteq R$.
17. Let R be a relation on a set A. Prove:
   (a) If R is reflexive, then $R \subseteq R^2$.
   (b) R is transitive iff $R^2 \subseteq R$.
18. Suppose that R and S are relations on a set A, where $R \subseteq S$ and S is transitive. Prove that $R^n \subseteq S$ for each positive integer n.
19. Suppose that R and S are symmetric relations on a set A. Prove:
   (a) If $(x,y) \in S \cdot R$, then $(y,x) \in R \cdot S$.
   (b) If $R \cdot S \subseteq S \cdot R$, then $R \cdot S = S \cdot R$.
   (c) $R \cdot S$ is symmetric iff $R \cdot S = S \cdot R$.
   (d) R^n is symmetric for each positive integer n.
20. Suppose R and S are relations on a set A. Prove or disprove:
   (a) If R and S are reflexive, then so is $R \cdot S$.
   (b) If R and S are both reflexive and symmetric, then $R \cdot S$ is reflexive and symmetric iff $R \cdot S = S \cdot R$.
   (c) If R and S are transitive, then $R \cdot S$ is transitive iff $R \cdot S = S \cdot R$.
   (d) If R and S are equivalence relations on A, then $R \cdot S$ is an equivalence relation iff $R \cdot S = S \cdot R$.
21. Suppose R and S are relations on a set A.
   (a) Prove that $(R \cdot S)^{-1} = S^{-1} \cdot R^{-1}$.
   (b) Is it true that $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$?
   (c) Is is true that $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$?
   (d) If R is an equivalence relation on A, is it true that R^{-1} is an equivalence relation on A?
   (e) If R satisfies any of the six properties of relations defined in Section 4.2, determine whether or not R^{-1} satisfies the same properties.
   (f) Suppose that $R \subseteq S$. Show that $R^{-1} \subseteq S^{-1}$.
22. Assume that R is a reflexive relation on a set A.
   (a) Show that $R \cdot R^{-1}$ is reflexive and symmetric.
   (b) Prove or disprove $R \cdot R^{-1}$ is transitive.
   (c) Prove or disprove that the transitive closure $R \cdot R^{-1}$ is an equivalence relation.
23. Let R and S be relations from A to B and let T and W be relations from B to C. Prove:
   (a) $R \cdot (T \cup W) = (R \cdot T) \cup (R \cdot W)$
   (b) $R \cdot (T \cap W) = (R \cdot T) \cap (R \cdot W)$
   (c) If $R \subseteq S$, then $R \cdot T \subseteq S \cdot T$.

nd Search and

aphs as models
Flow and Min-

Multigraphs
and Four-Color

7. Define the relation C on the vertices of a digraph such that $x\ C\ y$ iff there is a nondirected path from x to y. Prove or disprove:
   (a) C is an equivalence relation.
   (b) the equivalence classes of C are each weakly connected components of the digraph.

8. Give the definition of an equivalence relation on the vertices of a digraph such that the equivalence classes together with the edges between these vertices are the strongly connected components.

9. Define the relation D on the vertices of a digraph such that $x\ D\ y$ iff there is a directed path from x to y. Prove or disprove:
   (a) D is not an equivalence relation on the vertices of a digraph.
   (b) D is a partial order iff the digraph has no cycles of length greater than one.

10. Let $A = (V,E)$ be a digraph. Define $A^1 = (V,E^1)$ where $(x,y) \in E^1$ iff $(x,y) \in E$ or $(y,x) \in E$. Prove or disprove:
    (a) E^1 is an equivalence relation.
    (b) A^1 is unilaterally connected iff A is weakly connected.

11. Let $A = (V,E)$ be a digraph. Define $A^* = (V,E^*)$. Prove or disprove that A^* is strongly connected iff A is unilaterally connected.

12. Prove that the following definitions of E^n are equivalent:
    (a) $E^1 = E$ and $E^n = E^{n-1} \cdot E$ for $n > 1$;
    (b) $E^1 = E$ and $E^n = E \cdot E^{n-1}$ for $n > 1$.

13. Prove that the following definitions of E^* are equivalent:
    (a) $E^* = \bigcup_{i=1}^{\infty} E^i$;
    (b) $C^1 = E$, $C^{n+1} = C^n \cdot C^n$, and $E^* = \bigcup_{i=1}^{\infty} C^i$.

14. Let R be a relation on a set A and let $S = R^2$. Prove that $(x,y) \in S^+$ iff there is a directed path in R from x to y of even length.

15. Prove by induction: If P is a path from a vertex x to a vertex y, then P contains a simple path.

16. Prove that a digraph G is unilaterally connected iff there is a directed path in G containing all the vertices of G.

17. Prove that a digraph G is strongly connected iff there is a closed directed path containing every vertex in G. (A path (v_0,v_1), $(v_1,v_2),\ldots,(v_{n-1},v_n)$ is *closed* if $v_0 = v_n$.)

§ 3.6

18. Solve the following recurrence relations by making an appropriate substitution to transform the relations into linear recurrences with constant coefficients.
    (a) $\sqrt{a_n} - \sqrt{a_{n-1}} - 2\sqrt{a_{n-2}} = 0$ where $a_0 = a_1 = 1$.
    (b) $na_n + na_{n-1} - a_{n-1} = 2^n$ where $a_0 = 10$.
    (c) $a_n^3 - 2a_{n-1} = 0$ where $a_0 = 8$. Hint: let $b_n = \log_2 a_n$.
    (d) $a_n - na_{n-1} = n!$ for $n \geq 1$ where $a_0 = 2$.
    (e) $a_n = \dfrac{\sqrt{a_{n-1}}}{a_{n-2}^2}$ where $a_0 = 1$ and $a_1 = 2$.
    (f) $a_n + 5na_{n-1} + 6n(n-1)a_{n-2} = 0$ where $a_0 = 6$ and $a_1 = 17$.
    (g) $a_n = (a_{n-1})^2\,(a_{n-2})^3$ where $a_0 = 4$ and $a_1 = 4$.
    (h) $na_n = (n-2)a_{n-1} + 2n$ where $a_0 = 5$.
    (i) $na_n = (n+1)a_{n-1} + 2n$ where $a_0 = 1$.

24. Solve the divide-and-conquer relations using a change of variables.
    (a) $a_n = 5a_{n/2} - 4$ where $a_1 = 0$ and $n = 2^k$ for $k > 0$.
    (b) $a_n = 2a_{n/3} + 4$ where $a_1 = 5$ and $n = 3^k$ for $k > 0$.
    (c) $a_n = 3a_{n/8} + 2n$ where $a_1 = 1$ and $n = 8^k$ for $k > 0$.
    (d) $a_n = 5$ [illegible] n where $a_1 = 5/2$ and $n = 3^k$ for $k > 0$.

3. Applicant A is qualified for jobs J_1, J_2, and J_6; B is qualified for jobs J_3, J_4, J_5, and J_6; C is qualified for jobs J_1 and J_5; D is qualified for J_1, J_3, and J_6; E is qualified for J_1, J_2, J_4, J_6, and J_7; F is qualified for J_4 and J_6; and G is qualified for J_3, J_5, and J_7.
   (a) Model this problem as a matching network.
   (b) Find a maximal matching.
   (c) Is there a complete matching?
4. Five students have agreed to fix an old house as a church charity project. The house must be painted, wallpapered, and cleaned. In addition, furniture must be moved and new curtains made. Suppose that Anna can paint and move furniture; Jody can make new curtains; Brian can paint, wallpaper, and move furniture; Myrtie can clean and wallpaper; and Joe can paint and wallpaper. Assuming that each does a different job, give an assignment of tasks that will match as many people as possible with jobs that they can do.
5. A set of code words $\{bcd, aefg, abef, abdf, abc, cdeg\}$ is to be transmitted. Is it possible to represent each word by one of the letters in the word so that the words will be uniquely represented? If so, how?
6. Five students, S_1, S_2, S_3, S_4, and S_5, are members of four committees, C_1, C_2, C_3, and C_4. The members of C_1 are S_1, S_3, and S_4; C_2 has members S_3 and S_5; S_1, S_4, and S_5 belong to C_3; and S_1, S_2, S_3, and S_5 belong to C_4. Each committee is to send a representative to a banquet. No student can represent two committees.
   (a) Model this problem as a matching network.
   (b) What is the interpretation of a maximal matching?
   (c) What is the interpretation of a complete matching?
   (d) Find a maximal matching.
   (e) Is there a complete matching?
7. Six senators, S_1, S_2, S_3, S_4, S_5, and S_6, are members of 5 committees. The committees are $C_1 = \{S_2,S_3,S_4\}$, $C_2 = \{S_1,S_5,S_6\}$, $C_3 = \{S_1,S_2,S_5,S_6\}$, $C_4 = \{S_1,S_2,S_4,S_6\}$, and $C_5 = \{S_1,S_2,S_3\}$. The activities of each committee are to be reviewed by a senator who is not on the committee. Can 5 distinct senators be selected for the reviewing tasks? If so, how?
8. There are mn couples at a dance. The men are divided into m groups with n men in each group according to their ages. The women are also divided into n groups with m women in each group according to their heights. Show that m couples can be chosen so that every age and every height will be represented.
9. Six persons $P_1, P_2, \ldots, P_6$ are held in a foreign prison. The prison warden wishes to keep the prisoners separated into cells so that inmates cannot understand each other. Suppose that P_1 speaks

Chinese, French, and Hebrew; P_2 speaks German, Hebrew, and Italian; P_3 speaks English and French; P_4 speaks Chinese and Spanish; P_5 speaks English and German; and P_6 speaks Italian and Spanish. Could these 6 prisoners be locked in 2 cells such that no inmates in the same cell would be able to understand a language the others speak?
10. Suppose m applicants apply for n jobs where $m > n$. Without knowing anything about the qualifications, can you determine whether or not there is a complete matching? Explain.
11. If every girl in school has k boyfriends and every boy in school has k girlfriends, is it possible for each girl to go to the school dance with one of her boyfriends and for each boy to go to the dance with one of his girlfriends? Explain.
12. Give an example of a directed bipartite graph that has a complete matching but does not satisfy the conditions (i) and (ii) of Corollary 7.5.1.
13. Prove or disprove. Any matching is contained in a maximal matching.
14. There are n computers and n disk drives. Each computer is compatible with m disk drives and each disk drive is compatible with m of the computers. Is it possible to match each computer with a compatible disk drive?
15. An $r \times n$ Latin rectangle, where $r \leq n$, is an $r \times n$ matrix that has the numbers $1, 2, \ldots, n$ as entries such that no number appears more than once in the same row or the same column. A Latin square is an $n \times n$ Latin rectangle. If $r < n$ show that it is always possible to append $n - r$ rows to an $r \times n$ Latin rectangle L to form an $n \times n$ Latin square. Hint: do the following:
    (a) Let $A = \{a_1, \ldots, a_n\}$ denote the set of columns of L and let $B = \{1, \ldots, n\}$. Design a directed bipartite graph G with edges (a_i, k) where k does not appear in column a_i of L.
    (b) Show that there is a complete matching for the matching network determined by G.
    (c) Interpret what a complete matching means in this context.
16. Let G be a directed bipartite graph with vertex set V equal to the disjoint union of subset A and B where each edge of G is from vertices of A to vertices of B. Define the *deficiency of* G as $d(G) = \max\{|C| - |R(C)| \text{ such that } C \subseteq A\}$.
    (a) Show that G has a complete matching iff $d(G) = 0$.
    (b) Obtain the matching network (G^*, k) for G. Show that the capacity of any cut in (G^*, k) is greater than or equal to $|A| - d(G)$.