MAD 3105 — Discrete Math 2

Section 2, Fall 1993.

Instructor: Bellenot.

The good doctor's Office: 002-B Love, Office Hours: MW 1:30–2:20 or by appointment.

Eligibility: A grade of C- or better in Discrete 1 (MAD 3104)

Text: Mott, Kandel, Baker; 2nd Edition.

Coverage: Some or all of chapters 2, 3, 4, 5 and 7.

Final: At 7:30 – 9:30 am Thursday Dec 16, 1993.

Tests: (3) Tentatively at Sept 22, Oct 27(Nov 5?) and Dec 1. No Makeup tests.

Quizes: Every Wednesday (except test days). No Makeup quizes.

Grades: 90% A, 80%B, 70%C, 60%D.

Relative Weights F = 2T and T = Q (F is 1/3, each T is 1/6 and Q is 1/6).

Homework and Attendance are required.

Fair Warning: The good doctor teaches this course differently than other instructors. The order of material will be different and so will be the emphasis.

MAD 3105 Discrete Math 2 **Test 1** 22 Sept 1993

Show **ALL** work for credit; be neat; and use only **ONE** side of each page of paper.

1. Write the general solution to the homogenuous linear recurrence relation if the characteristic polynomial is
   A. $x^2 - 5x + 6$.
   B. $x^2 + 4x + 4$.
   C. $x^2 - 2x + 1$.
   D. $x^2 - x - 1$.

2. Give the *correct guesses* to the form of the particular solution of the linear non-homogenuous recurrence relation which has the given forcing function. The characteristic polynomial is known to have roots: -5, 1, 1, 3, 3, 3, 3, 5 and 6.
   A. $145 \cdot 2^n$.
   B. $(-n^2 + 7n - 5)4^n$.
   C. $65 \cdot 5^n$.
   D. $n^2 + 7$.

3. Show that the substitution $b_n = \log_2(a_n)$ transforms the non-linear recurrence relation $a_n = (a_{n-1})^2(a_{n-2})^3; a_0 = 4, a_1 = 8$, into a linear recurrence relation. Do **NOT** solve these recurrence relations. **Do** translate the initial conditions.

4. **Simplify** the following to the form n^α.
   A. $7^{\log_3 n}$.
   B. $(7^\pi)^{\log_5 n}$.

5. Show $a_n = n\log_2(n)$ is a solution to $a_n = 2a_{n/2} + n, a_1 = 0$; for $n = 2^k \geq 1$.

6. Solve: $a_n - a_{n-1} - 6a_{n-2} = 0; a_0 = 2, a_1 = 6$.

7. Solve: $a_n - a_{n-1} = n + 1; a_0 = 7$.

8. Solve by **GENERATING FUNCTIONS:** $a_n = 5a_{n-1} + 3^n, n \geq 1; a_0 = 2$.

MAD 3105 Discrete Math 2 **Test 1 & ANSWERS** 22 Sept 1993

Show **ALL** work for credit; be neat; and use only **ONE** side of each page of paper.

1. Write the general solution to the homogenuous linear recurrence relation if the characteristic polynomial is
   A. $x^2 - 5x + 6$. *Answer:* $A2^n + B3^n$
   B. $x^2 + 4x + 4$. *Answer:* $A(-2)^n + Bn(-2)^n$
   C. $x^2 - 2x + 1$. *Answer:* $An + B$
   D. $x^2 - x - 1$. *Answer:* $A((1-\sqrt{5})/2)^n + B((1+\sqrt{5})/2)^n$

2. Give the *correct guesses* to the form of the particular solution of the linear non-homogenuous recurrence relation which has the given forcing function. The characteristic polynomial is known to have roots: -5, 1, 1, 3, 3, 3, 3, 5 and 6.
   A. $145 \cdot 2^n$. *Answer:* $A2^n$
   B. $(-n^2 + 7n - 5)4^n$. *Answer:* $(An^2 + Bn + C)4^n$
   C. $65 \cdot 5^n$. *Answer:* $nA5^n$
   D. $n^2 + 7$. *Answer:* $n^2(An^2 + Bn + C)$

3. Show that the substitution $b_n = \log_2(a_n)$ transforms the non-linear $a_n = (a_{n-1})^2(a_{n-2})^3$; $a_0 = 4, a_1 = 8$, into a linear recurrence relation. Do **NOT** solve these recurrence relations. **Do** translate the initial conditions.

Answer: First the initial conditions: $b_0 = \log_2 a_0 = \log_2 4 = 2$ and $b_1 = \log_2 a_1 = \log_2 8 = 3$. Second, taking $\log_2$ of both sides of the original recurrence relation yields $\log_2 a_n = 2\log_2 a_{n-1} + 3\log_2 a_{n-2}$ or $b_n = 2b_{n-1} + 3b_{n-2}$ which is a linear recurrence relation.

4. **Simplify** the following to the form n^α.
   A. $7^{\log_3 n}$. *Answer:* $n^{\log_3 7}$
   B. $(7^\pi)^{\log_5 n}$. *Answer:* $n^{\pi \log_5 7}$

5. Show $a_n = n\log_2(n)$ is a solution to $a_n = 2a_{n/2} + n, a_1 = 0$; for $n = 2^k \geq 1$.

Answer: First the initial condition $a_1 = 1\log_2 1 = 1 \cdot 0 = 0$, which is correct. Second substituting we get see $RHS = 2a_{n/2} + n = 2(n/2)\log_2(n/2) + n = n(\log_2 n - \log_2 2) + n = n(\log_2 n - 1) + n = n\log_2 n - n + n = n\log_2 n = a_n = LHS$.

6. Solve: $a_n - a_{n-1} - 6a_{n-2} = 0; a_0 = 2, a_1 = 6$.

Answer: The general solution is $a_n = A(-2)^n + B3^n$. Using the initial conditions we get $2 = a_0 = A + B$ and $6 = a_1 = -2A + 3B$. Adding twice the first equation to the second yields $10 = 5B$ so $B = 2$ and $A = 0$. The answer is $a_n = 2 \cdot 3^n$.

7. Solve: $a_n - a_{n-1} = n + 1; a_0 = 7$.

Answer: First the particular solution has form $a_n = (An + B)n = An^2 + Bn$. Substituting $An^2 + Bn - A(n-1)^2 - B(n-1) = n^2(A - A) + n(B + 2A - B) + 1(-A + B)$ so we solve $2A = 1$ and $B - A = 1$ so $A = 1/2, B = 3/2$.. Thus the general solution is $a_n = C + n^2/2 + 3n/2$ and when $n = 0, 7 = a_0 = C + 0 + 0$, so $C = 7$ and $a_n = 7 + 3n/2 + n^2/2$.

8. Solve by **GENERATING FUNCTIONS:** $a_n = 5a_{n-1} + 3^n, n \geq 1; a_0 = 2$.

Answer: $A(X) = \sum_{n=0}^{\infty} a_n X^n = a_0 + \sum_{n=1}^{\infty} a_n X^n = 2 + \sum_{n=1}^{\infty}(5a_{n-1} + 3^n)X^n = 2 + 5X\sum_{n=1}^{\infty} a_{n-1}X^{n-1} + 3X\sum_{n=1}^{\infty} 3^{n-1}X^{n-1} = 2 + 5X\sum_{n=0}^{\infty} a_n X^n + 3X\sum_{n=0}^{\infty} 3^n X^n = 2 + 5XA(X) + \frac{3X}{1-3X}$ Solving for $A(X)$ yields

$$A(X)(1-5X) = \frac{2-6X}{1-3X} + \frac{3X}{1-3X}. \; A(x) = \frac{2-3X}{(1-5X)(1-3X)} = \frac{B}{1-5X} + \frac{C}{1-3X}.$$

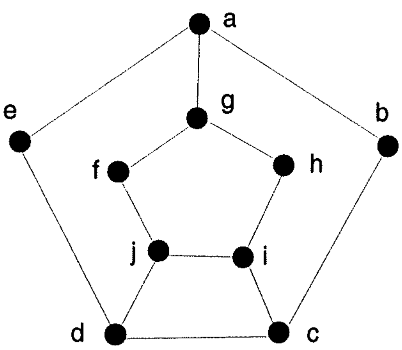
$$B = \frac{2-3(1/5)}{1-3(1/5)} = \frac{7/10}{2/5} = \frac{7}{4}. \; C = \frac{2-3(1/3)}{1-5(1/3)} = \frac{1}{-2/3} = \frac{-3}{2}$$

Thus $A(x) = (7/4)\sum_{n=0}^{\infty} 5^n + (-3/2)\sum_{n=0}^{\infty} 3^n$ and hence $a_n = (7/4)5^n + (-3/2)3^n$.

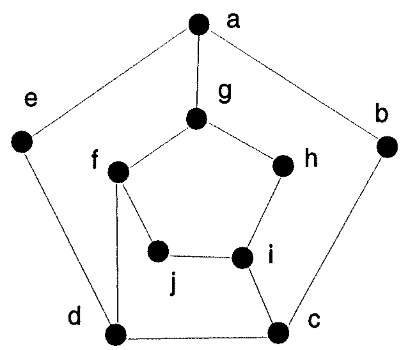
MAD 3105 Discrete Math 2 **Test 2** 5 Nov 1993

Show **ALL** work for credit; be neat; and use only **ONE** side of each page of paper.

1. What big O (i.e. $O(f(n))$) best describes the run time of the following algorithms:
   A. Sorting n items by Merge Sort.
   B. Searching in a sorted array of n items by Binary Search.
   C. Sorting n items by Interchange Sort.
   DE. Where the run time is given by $a_n = 4a_{n/2}; a_1 = 1$. Simplify.



Graph 1



Graph 2

2. For Graph 1 above:
   A. Either construct an Euler circuit or prove none exists.
   B. Either construct a Hamiltonian cycle or prove none exists.
   C. Draw the dual graph.

3. Disorder and more big O.
   A. Prove $O(4n^2 - 5 + 13n^3) = O(100n^3 - n + n^2 - 3)$.
   B. Arrange in increasing order:

$$O(\log n), O(n!), O(n^6), O(\sqrt{n}), O(3^n), O(n), O(n\sqrt{n}), O(n^n), O(n \log n), O(2^n).$$

   C. Compute the disorder of 2, 7, 3, 1, 4, 5, 9, 8, 6.

4. Show that Graph 2 above does not have a Hamiltonian cycle:
   A. Using Grinberg's Theorem.
   B. Without using Grinberg's Theorem.

5. For the relation R on the set of integers defined by $xRy \iff 0 < |x - y| < 3$. Either prove R has the given property or give a counter-example to show it does not have that property.
   A. Reflexive.
   B. Symmetric.
   C. Anti-symmetric.
   D. Transistive.

6. Mod m problems.
   A. Find $0 \leq x < 31$ so that $7x \equiv 1 \bmod 31$
   B. Find $0 \leq x < 5$ so that $3^{101} \equiv x \bmod 5$
   C. What is the remainder if $\sum_{i=1}^{100} i^2$ is divided by 5.

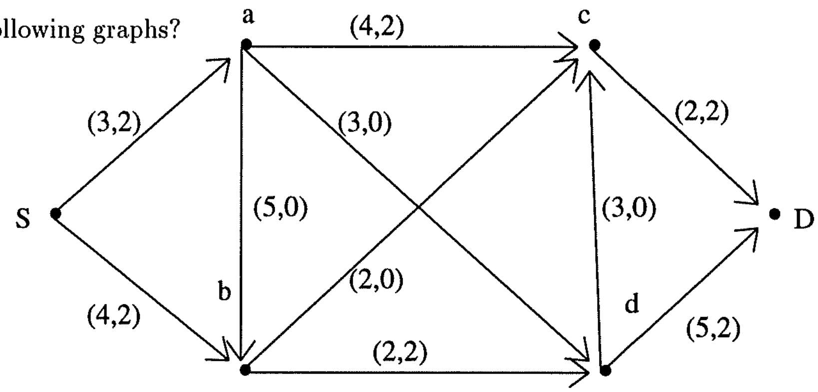
7. Prove the following relation R is an equivalence relation and describe the equivalence class of (3,4). The relation R is defined on the points of the plane by $(a, b)R(c, d) \iff a^2 + b^2 = c^2 + d^2$.

8. Use Euler's formula and $2|E| \geq 3|R|$ to show $|E| \leq 3|V| - 6$. Then show a plane connected (simple) graph with strictly fewer than 30 edges has a vertex of degree 4 or less.

MAD 3105 Discrete Math 3 **Test 3** 3 Dec 1993

Show **ALL** work for credit; be neat; and use only **ONE** side of each page of paper.

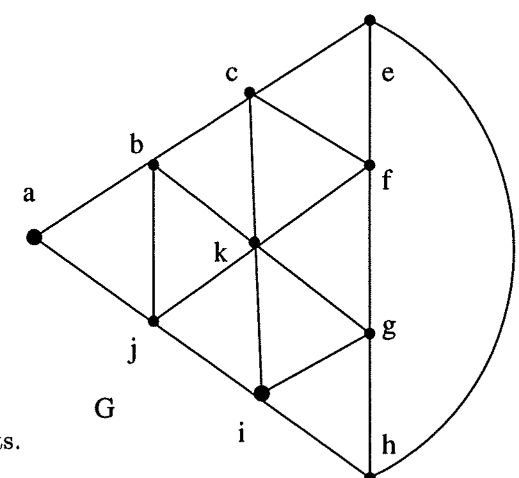
1. What is the chromatic number of the following graphs?
   A. K_n
   B. $\overline{K_n}$
   C. $C_n, n \geq 4, n$ even.
   D. $C_n, n \geq 3, n$ odd.
   E. A non-trivial tree.



2. For the transport network above use our labeling algorithm to find a maximal flow, its value, and a minimal cut.

3. Network True or False.
   A. If $F(X,\overline{X}) - F(\overline{X},X) = k(X,\overline{X})$, then F is a maximal flow.
   B. If $F(\overline{X},X) = 0$ and $(X,\overline{X})$ is minimal cut, then F is a maximal flow.
   C. A cut can be minimal for one flow, and not minimal for another flow.
   D. If ab is an unsaturated edge with non-zero flow and $(X,\overline{X})$ is a minimal cut, then either both vertices are in X or both vertices are in $\overline{X}$.
   E. If $|F| = 0$, then every edge has zero flow.

4. Give network counterexamples to each statement below:
   A. A transport network with a unique maximal flow has a unique minimal cut.
   B. A transport network with a unique minimal cut has a unique maximal flow.
   C. If F is a flow and $(X,\overline{X})$ is a cut so that $F(\overline{X},X) = 0$,then $(X,\overline{X})$ is a minimal cut.



5. For G above
   A. Find an Euler circuit or prove none exists.
   B. Find an Euler path or prove none exists.
   C. Find an Hamiltonian cycle or prove none exists.

6. Find $\chi(G)$ for G above. Prove G has this chromatic number.

7. Using $2|E| \leq 3|R|$ and Euler's formula show plane connected (simple) graphs satisfies $|E| \leq 3|V| - 6$. Show plane connected (simple) graphs have a vertex of degree 5 or less.

8. There are exactly two 3-regular graphs with 6 vertices, $K_{3,3}$ and the "trianglar prism". For both of these graphs, either give a plane drawing of the graph, or carefully prove the graph is non-planar.