DM2 ~~MAD~~ 3105 -01 PROF BELLENOT

FALL 86

218 LOVE

MWF 2:30 - 3:30

(The good doctor is available most afternoons (if he is in))

TEXT: MOTT, KANDEL, BAKER "Discrete Math for ..."
(Is it the 2nd Edition?) Chapts 3, 4, 7 & rest of 2 & 5

IMPORTANT DATES:

FINAL @ 3:00pm MON 8 DEC.

MIDTERMS 1 OCT & 5 NOV (TENTATIVE)

GRADES: 90, 80, 70, 60 percentage cut offs based on FINAL 36% MIDTERMS (18% EACH) TP's (18%) AND HW 10%.

HOMEWORK: HW is given each class to be turned in the next class. Late HW isn't accepted. Your HW grade is ~~based~~ on min (100%, HW done / .90 × HW assigned)

TP's: Homework which is carefully graded on a 0-10 ~~base~~ basis. Assigned a week in advance of its due date. (Only the top 2/3's count towards your TP score.)

YOU NEED A "C-" or better in MAD 3104 to take this course.

DM2 PT 1 1-4 10 pts 5-8 15 pts * problems are from old tests ** problems are "close" to old tests

1. *A. In Z_{13} find i with $0 \le i < 13$ so that $[i] = \frac{[2]}{[9]}$

**B. In the ordering $\mathbb{N}, |$ ($|$ = divides) find lub(1462, 132)

*2. The roots of ~~a~~ the characteristic poly are 2, 2, 1, 5 ~~write~~

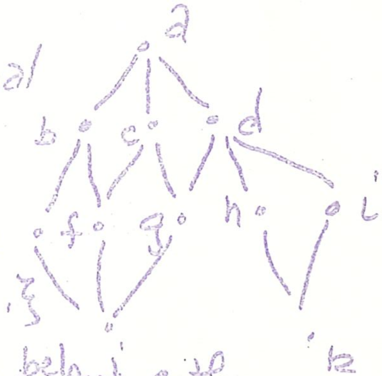
A. write the general solution ~~to~~ the homo prob.

B-E for the given forcing function f(n) give the correct guess for the particular solution

B. $f(n) = 3 \cdot 4^n$ C $f(n) = n^2 + 7$ D. $f(n) = \frac{1}{2} \cdot 5^n$ E. $f(n) = 7 \cdot 2^n$

*3. For the p.o. "graph" to right

A list all maximal elements B list all minimal elements. C list all elements "less than or equal to d D glb(b,g) E lub(j,h)



*4 The relation R is defined on {1, 2, 3, ...} by $x R y \iff x \mid (y+1)$. For each property below either say yes or say no and give a counterexample

A. ref B. irref. C. trans. D anti-sym

*5. Solve $a_n + a_{n+1} = 6a_{n+2}$ $a_0 = 5$ $a_1 = 25$

6. Give counterexamples

*A an equivalence relation on a set with 2 or more elements is never a p.o. (draw digraph)

**B a relation which is not reflexive is irreflexive

C. if R & S are symmetric relations then $S \circ R$ is symmetric

D. two strings have the same relationship in both the lexicographic and the enumeration orderings.

E. If the relation $R \subseteq R^2$ then R is transitive.

*7. K_n (complete graph on n vertices) [not digraph] has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges. Prove this by induction.

8. We have proved that each finite p.o. set has a maximal element. Prove that a finite p.o. set with only one maximal element has a greatest element!

MORE PROBLEMS FROM OLD TESTS

10 pt problems

1. $a_0 = 0$, $a_n = 2a_{n-1} + 2^{n-2}$ for $n \ge 1$. prove by induction $a_n = 4n2^n$ $n \ge 0$

2. $a_1 = 2$, $a_n = \frac{1}{a_{n-1}} + \frac{2}{n}$ for $n \ge 2$, prove by induction $a_n = \frac{n+1}{n}$ $n \ge 1$

3. $(x+1)(x-3)^2 = x^3 - 5x^2 + 3x + 9$. For the recurrence relation
$a_n - 5a_{n-1} + 3a_{n-2} + 9a_{n-3} = f(n)$
AB: write the gen sol when $f(n) = 0$. CDE write the correct guess for the particular sol when $f(n) = 4n^2+1$, $D = 6 \cdot 3^n$ E $2n(-1)^n$
(do not solve for the part. sol.)

15 pt problems

4. $a_n - 7a_{n-1} + 12a_{n-2} = 12n - 4$ $a_0 = 5$ $a_1 = 8$

5. $a_0 = 3$ $a_1 = 4$ $a_n = 4a_{n-1} - 4a_{n-2}$ $(n \ge 2)$
Prove by induction $a_n = 3 \cdot 2^n - n2^n$ $n \ge 0$

---

other Problems

A. Define lattice, total order, well order, equiv rel...

B. Give examples: 1. total order which isn't a well order 2. a lattice which isn't totally ordered. 3. A partially ordered set which isn't a lattice.

C. Give counterexamples: 1. lexicographic order is a well ordering. 2. if a subset B of a p.o. set A has an upper bound then B has a greatest element 3. $B \subset A \le$ has a upper bound then it has a lub.

D. Prove:

1. If $R^2 \subset R$ then R is trans (R is a relation)

2. If R is symmetric, transitive and has the property that $\forall x, y$ then xRy or yRx, then R is a total order.

3. If R is symmetric, prove by induction that R^n is symmetric for $n \ge 1$

MD2

TP2 due Fri 5 Sept

The relation R is defined on the set $\mathbb{R}$ of all real numbers by $xRy \iff x+y \geq -10$.

For each of the following statements provide either a proof or disprove the statement (i.e. give a counterexample)

A. R is transitive

B. R is reflexive

C. R is irreflexive

D. R is symmetric

E. R is anti-symmetric

F. R is Asymmetric

TP 3 due Mon 8 Sept

Given $a_0 = \sqrt{2}$ & for $n \geq 1$ $a_n = \sqrt{2 + a_{n-1}}$

Prove by induction for $n \geq 0$ $a_n < 2$.

TP 4 due Wed 10 Sept

A. Given R is a partial order on the set A and $\acute{S}R$ is defined by

$$x \acute{S}R\, y \iff xRy \ \& \ x \neq y$$

Show the relation $\acute{S}R$ ("Strictly R") is irreflexive, asymmetric and transitive

B Given R is a irreflexive, asymmetric and transitive relation on A and RE is defined by

$$x\, RE\, y \iff xRy \text{ or } x = y$$

Show the relation RE ("R or equal") is a partial order on A.

TP 5 due Fri 12 Sept.

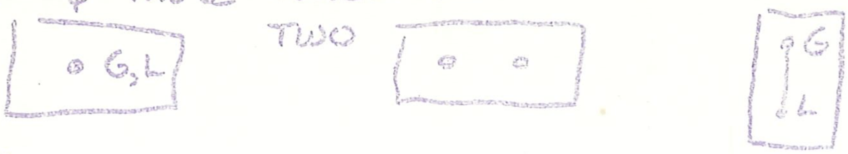
Given $a_0 = 10$ and $a_1 = 0$ and for $n \geq 2$

$$a_n + a_{n-1} - 6a_{n-2} = 0$$

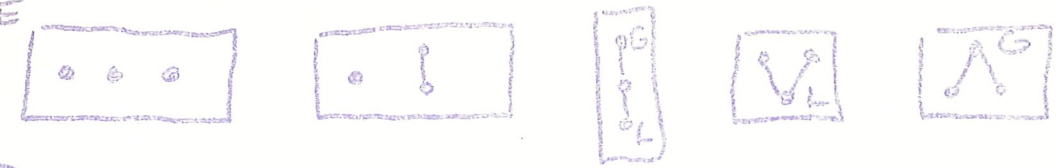
Prove by induction $a_n = 6 \cdot 2^n + 4 \cdot (-3)^n$ for $n \geq 0$

A TP & more answers

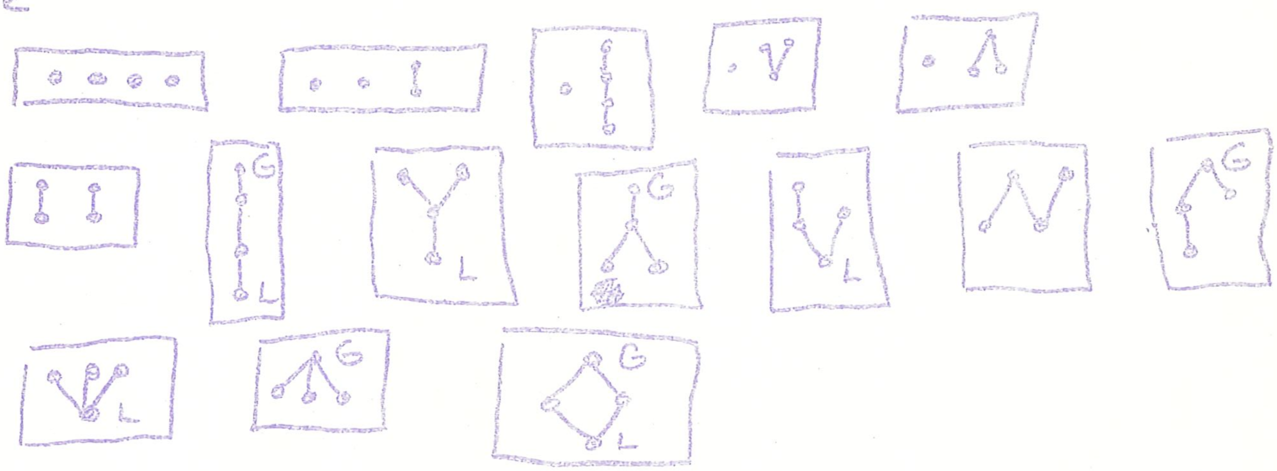
23. ONE TWO



THREE



FOUR



20 a. $\mathbb{R}, \leq$ ($\mathbb{R}$ itself has no least element also $\{x \in \mathbb{R} : 0 < x\}$

b.  ... etc

c. $\mathbb{R}, \leq$ $\mathbb{R}$ or $\{x \in \mathbb{R} : x < 0\}$

d. $\mathbb{R}, \leq$ $B = \{x \in \mathbb{R} : x < 0\}$ has lub 0 but $0 \notin B$

e. $\mathbb{Q}, \leq$ (rationals) $B = \{x \in \mathbb{Q} : x < \sqrt{2}\}$

the upper bounds to B are $U = \{x \in \mathbb{Q} : x > \sqrt{2}\}$

(since $\sqrt{2}$ is irrational) and U has no least element.

TP 7 Due 19 Sept. For $n \geq 1$ $a_n = (-1)^{n+1} \frac{1}{n}$ and $S_n = \sum_{i=1}^{n} a_i$. PROVE for n even ≥ 2

$S_n < S_{n+2} < S_{n+1}$ and for n odd ≥ 1

$S_n > S_{n+2} > S_{n+1}$, By induction.

TP8 due mon 22 sept

Given: R is a ~~relation~~ partial order on A. Relations S and T are defined on A as follows

$$xSy \iff xRy \text{ and } x \neq y$$

$$xTy \iff xSy \text{ or } x = y.$$

[We already know that T is a partial order on A.]

Prove $R = T$

TP9 due wed 24 sept

Given $a_0 = 5$ $a_1 = 16$ & for $n \geq 2$ $a_n = 4a_{n-1} - 4a_{n-2} + 6 \cdot 2^n$

$$(a_n = 4a_{n-1} - 4a_{n-2} + 6 \cdot 2^n)$$

Prove by induction $a_n = 5 \cdot 2^n + 3n^2 \cdot 2^n$ for $n \geq 0$

TP 11. Due Mon 6 Oct

PROVE OR DISPROVE

A. FOR ANY RELATION R WE HAVE EITHER $R \subseteq R^2$ OR $R^2 \subseteq R$.

B. FOR ANY RELATION R WE HAVE $(R^{-1})^2 = (R^2)^{-1}$

TP 12 Due Wed 8 Oct

PROVE BY INDUCTION, If the relation R is reflexive and transitive, then $R^n = R$ $(n \geq 1)$

DM2

TP 13 due Mon 13 Oct.

~~Proof~~. A. Given x & y are vertices ($x \neq y$) in a directed graph G with a directed path P from x to y and a directed path Q from y to z. Prove that G has a directed cycle.

B. Given x & y are vertices ($x \neq y$) in a (non-directed) graph G and P & Q are two different paths from x to y. Prove that G has a (non-directed) cycle.

TP 14 due Wed 15 Oct. [3.4 9, 10 by induction]

A. Given $n = d^k$ $k \geq 0$ $a_1 = c$

and for $k \geq 1$ $a_n = a_{n/d} + c$

Prove for $n = d^k$, $k \geq 0$

$$a_n = c(\log_d n + 1)$$

B Given $n = d^k$ $k \geq 0$ $a_1 = e$

and for $k \geq 1$ $a_n = c a_{n/d} + e$ $(c \neq 1)$

Prove for $n = d^k$, $k \geq 0$

$$a_n = e(c\, n^{\log_d c} - 1)/c - 1$$

DM2

TP 15 due mon 20 OCT

~~Prove or Disprove:~~

Prove or Disprove

Given G is connected.

A. x is a cut node of G $\Rightarrow$ ~~there~~ for all vertices y and z in G with $y \neq z$, there is a simple path P from y to z which traverses x.

Definition: A vertex x of a graph G is said to be a cut-node, if the graph H is disconnect where H is G with x and x's incident edges removed

B. For all vertices y and z in G with $x \neq y, y \neq z, z \neq x$, there is a simple path P from y to z which traverses x $\Rightarrow$ x is a cut node of G

TP 16 due wed 22 oct.

Prove or Disprove

A. If P is a simple path of maximal length from x to y in G and Q is a portion of P which goes from w to z, then Q is a simple path of maximal length from w to z in G.

B. If P is a simple path of minimal length from x to y in G and Q is a portion of P which goes from w to z, then Q is a simple path of minimal length from w to z in G.

DM2.

TP 17 due Mon Oct 27

[Helpful hint each even number can be written $2^k p$ where $k \geq 1$ and p is odd]

Given for n odd ≥ 1, $a_n = n \log_2 n$ and for n even ≥ 2, $a_n = 2a_{n/2} + n$.

Prove by induction (on k in helpful hint) for n even ≥ 2 $a_n = n \log_2 n$

TP 18 due Wed Oct 29

Prove or disprove:

A. A maximal directed path in a digraph G must be a directed path of maximal length in G

B. A minimal directed path in a digraph G must be a directed path of ~~maximal~~ minimal length in G.

TP19 was a "prove or disprove" if the following algorithms were correct solutions to TSP <traveling salesman Problem>

A. Greedy take the next shortest step possible

B. Divide & Conquer. Split in half & find "best" way of glueing together.

DM2 TP 20 due Mon 10 Nov

Prove or Disprove

A. If G is a transport network with the property that every edge e in G has non-zero capacity, then G has a non-zero flow.

B. All F-augmenting paths are simple (F is a flow on transport network)

TP 21 due Wed 12 Nov

A. If the digraph G has a closed directed path P which visits each vertex of G, then G is strongly connect.

B. If G is a digraph, let A be the collection of all subsets S of V(G) (vertices of G) so that there is a closed directed path P which visits each vertex in S. Now A ⊆ is poset, prove A has a maximal element T.

C. If the digraph G is strongly connected in B above then T = V(G).

DM 2

TP 22 due Mon 17 Nov.

Given G is a digraph with vertices $\{1, 2, \ldots n, n+1\}$ there is a path P in G which goes $1 \to 2 \to 3 \to \ldots \to n-1 \to n$ and for each $1 \le i \le n$ there is either an edge i to $n+1$ or an edge $n+1$ to i but <u>not</u> <u>both</u>!

<u>Prove</u>: There is a simple directed path Q in G which "visits" each vertex in G.

TP 23 due Wed 19 Nov

Problem 15 <u>Section 7.4</u>

DM2 .

TP 24 due Mon 24 Nov.

Given $\begin{bmatrix} a_0 = 6 \\ b_0 = 9 \end{bmatrix}$ and for $n \geq 1$ $\begin{bmatrix} a_n = 2a_{n-1} + 2b_{n-1} \\ b_n = -2a_{n-1} + 7b_{n-1} \end{bmatrix}$

Prove by induction for $n \geq 0$ that

$$\begin{bmatrix} a_n = 2 \cdot 3^n + 4 \cdot 6^n \\ b_n = 3^n + 8 \cdot 6^n \end{bmatrix}$$

TP 25 due Wed 26 Nov

Solve by generating fcns $a_0 = 2$ $b_0 = 3$

and for $n \geq 1$ $\begin{bmatrix} a_n = 19a_{n-1} - 3b_{n-1} \\ b_n = -3a_{n-1} + 11b_{n-1} \end{bmatrix}$

DM2 — the ~~last~~ 2 tp's !

TP 26 due 1 Dec Mon

A round-robin graph with n players is a digraph G on n vertices (one for each player. and for each pair of players (vertices) $1 \le i < j \le n$ $(i \ne j!)$ there is either an edge $i \to j$ or an edge $j \to i$ but not both. [$a \to b$ means a "beat" b !]

Prove by induction: A round-robin graph has a simple directed path which contains (visits) all the vertices.

TP 27 due 3 Dec Wed

Prove a digraph G is unilaterally connected if and only if there is a directed path P which visits every vertex of G

12 Sept 86 Some Answers: e

11

	upper bounds	lub	lower bounds	glb
c	NONE	NONE	1	1
d	NONE	NONE	1	1
e	error $10 \notin D_{12}$ 12	if $B=\{2,6\}$ 12	NONE	NONE
f	NONE	NONE	1	1

14. A. Reflexive: $x \le x$ & $y \le y$ hence $(x,y) \sqsubseteq (x,y)$.
Thus $\sqsubseteq$ is reflexive

Anti-Symmetry: if $(x,y) \sqsubseteq (w,z)$ & $(w,z) \sqsubseteq (x,y)$
then $x \le w$, $y \le z$, $w \le x$ & $z \le y$. Thus $x=w$ & $y=z$
Hence $(x,y)=(w,z)$ and $\sqsubseteq$ is anti-sym.

transitive: if $(x,y) \sqsubseteq$ [illegible] $s) \sqsubseteq (w,z)$ then
$x \le r$, [illegible] ence $x \le w$ & $y \le z$
and $(x,y$ [illegible]

NOT TOTAL ORDER [illegible] $0)$ nor $(1,0) \sqsubseteq (0,1)$
is true

B. (a) lub ($(x,$ [illegible] $w\}$, $\max\{y,z\})$
glb ($(x,$ [illegible] $w\}$, $\min\{y,z\})$

To see that [illegible]: Note that
$(\max\{x,w\}$, max [illegible] $)$ is an upper bound to $\{(x,y),(w,z)\}$
since $x, w \le \max\{x,w\}$, $y, z \le \max\{y,z\}$. To see
that it is the lub, let (r,s) be any other upper
bound. Thus $(x,y) \sqsubseteq (r,s)$ & $(w,z) \sqsubseteq (r,s)$ and
hence $x, w \le r$ and $y, z \le s$ or $\max\{x,w\} \le r$ &
$\max\{y,z\} \le s$. Therefore $(\max\{w,x\}, \max\{y,z\}) \sqsubseteq (r,s)$
and it is lub.

The glb. is done similarly.

JUNK

A. R is po so its REF, ANTI-SYM & TRANS

$xSRy \iff xRy$ & $x \neq y$

<u>SR is irreflexive</u>: since $x = x$ is always true, both $x \neq x$ and xRx & $x \neq x$ is always false. ∴ $x \not{S}Rx$

<u>SR is asymmetric</u>: Suppose not. Then $xSRy$ and $ySRx$, hence xRy, $x \neq y$, yRx & $x \neq y$. But R is anti sym, so xRy & $yRx \Rightarrow x = y$, an contradiction.

<u>SR is transitive</u>: If $xSRy$ & $ySRz$ then xRy, $x \neq y$, yRz and $y \neq z$. If $x = z$ then xRy & yRx are true (by replacing z with x); but then $x = y$ by R's anti-sym which contradicts $x \neq y$. Thus $x \neq z$ and since R is trans $xRz \Rightarrow xSRz$.

R is irref, asym & trans

$xREy \iff xRy$ or $x = y$. Show RE is p.o.

<u>RE is reflexive</u>

since $x = x$ is always true $x = x$ or xRx is also always true ∴ $xREx$

<u>RE is anti-sym</u>

suppose not then there are $x \neq y$ with both $xREy$ & $yREx$. Since $x \neq y$, $xREy \Rightarrow xRy$ and $yREx \Rightarrow yRx$ But this contradicts R's asymmetry.

<u>RE is trans</u>: Suppose $xREy$ & $yREz$. $xREy \iff xRy$ or $x = y$. If $x = y$, then we can replace y by x and get $xREz$. So assume $x \neq y$. If $y = z$, then we can replace y by z and get $xREz$. So assume $y \neq z$. Hence both xRy & yRz are true. But since R is trans xRz and $xREz$ are true.

TP 8: Proof: There are two cases to consider either $x=y$ or $x\neq y$

if $x=y$ then since T & R are reflexive xTx & xRx

hence "$xTy \Leftrightarrow xRy$" is true in this case

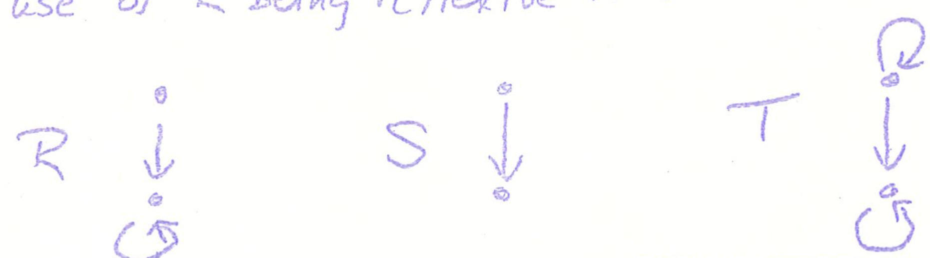
else $x\neq y$ now $xTy \Leftrightarrow xSy$ or $x=y \Leftrightarrow xSy$

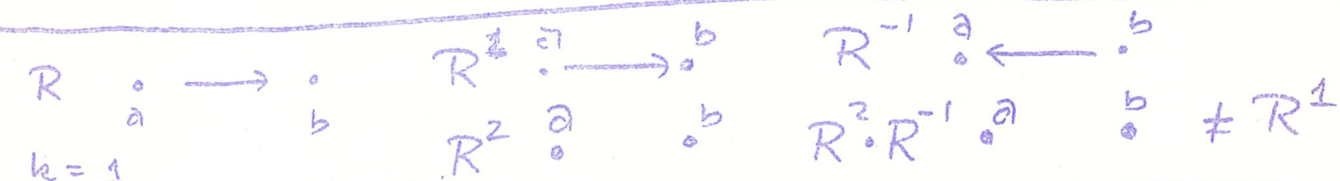
and $xRy \Leftrightarrow xRy$ and $x\neq y \Leftrightarrow xSy$

hence $xTy \Leftrightarrow xRy$ in this case too

$\therefore xTy \Leftrightarrow xRy$. And $R=T$

EXAMPLE: (shows that R must be a p.o. for TP8 to work, that is the use of R being reflexive is essential to the proof.)

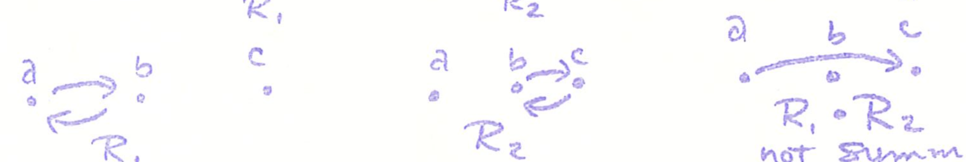




14 A. let $x \in A$. xR_1x & $xR_2x \Rightarrow xR_1\circ R_2x$ $\therefore R_1\circ R_2$ is ref.

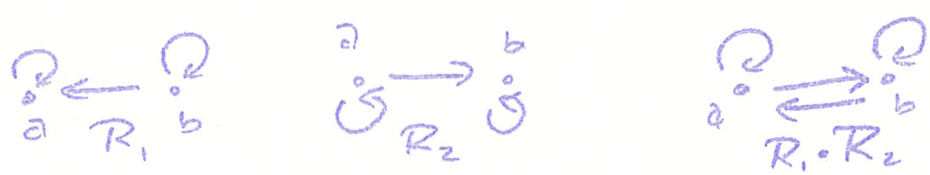
B. Counter example

C ~~Suppose~~



not symmetric

D.



not anti-symmetric

E.

not trans.

More on lattices. A poset $A \leq$ is a lattice if each pair of elements has a g.l.b. and l.u.b. Sometimes it is easier to write $\text{glb}\{x,y\} = z$ as $z = x \wedge y$ and $\text{lub}\{x,y\} = w$ as $w = x \vee y$.

Examples 1. $\mathcal{P}(X), \subseteq$ is a lattice. Think of x & y as ~~subsp~~ subets of X. $x \vee y$ is $x \cup y$ and $x \wedge y$ is $x \cap y$.

$x \cap y$ is a lowerbdd to $\{x,y\}$ because $x \cap y \subseteq x$ & $x \cap y \subseteq y$

$x \cap y$ is the glb since in addition if $z \subseteq x$ & $z \subseteq y$ then $z \subseteq x \cap y$.

2. $\mathbb{N}, |$ is a lattice ($|$ is the relation "divides")

Think of x, y as integers ≥ 1. $x \vee y$ is the ~~g.c.d.~~ l.c.m., the ~~greatest~~ least common multiple, the ~~biggest~~ smallest number z that ~~divides~~ both x & y divide into z. $x \wedge y$ is the g.c.d. the greatest common divisor. the largest number that divides both x & y.

3. (From Digital) The switching algebra $\{0,1\} \leq$ x & y are either 0 or 1 $x \vee y$ is written $x+y$ and $x \wedge y = xy$. [See digital for details]

4. Boolean Algebras [similar to 3 above]

5. $\mathbb{R} \leq$ {reals with the usual stuff}

$x \wedge y = \min\{x,y\}$. $x \vee y = \max\{x,y\}$ (Totally ordered sets are lattices.)

7. Define the relation C on the vertices of a digraph such that $x\,C\,y$ iff there is a nondirected path from x to y. Prove or disprove:
   (a) C is an equivalence relation.
   (b) the equivalence classes of C are each weakly connected components of the digraph.

8. Give the definition of an equivalence relation on the vertices of a digraph such that the equivalence classes together with the edges between these vertices are the strongly connected components.

9. Define the relation D on the vertices of a digraph such that $x\,D\,y$ iff there is a directed path from x to y. Prove or disprove:
   (a) D is not an equivalence relation on the vertices of a digraph.
   (b) D is a partial order iff the digraph has no cycles of length greater than one.

10. Let $A = (V,E)$ be a digraph. Define $A^1 = (V,E^1)$ where $(x,y) \in E^1$ iff $(x,y) \in E$ or $(y,x) \in E$. Prove or disprove:
    (a) E^1 is an equivalence relation.
    (b) A^1 is unilaterally connected iff A is weakly connected.

11. Let $A = (V,E)$ be a digraph. Define $A^* = (V,E^*)$. Prove or disprove that A^* is strongly connected iff A is unilaterally connected.

12. Prove that the following definitions of E^n are equivalent:
    (a) $E^1 = E$ and $E^n = E^{n-1} \cdot E$ for $n > 1$;
    (b) $E^1 = E$ and $E^n = E \cdot E^{n-1}$ for $n > 1$.

13. Prove that the following definitions of E^* are equivalent:
    (a) $E^* = \bigcup_{i=1}^{\infty} E^i$;
    (b) $C^1 = E$, $C^{n+1} = C^n \cdot C^n$, and $E^* = \bigcup_{i=1}^{\infty} C^i$.

14. Let R be a relation on a set A and let $S = R^2$. Prove that $(x,y) \in S^+$ iff there is a directed path in R from x to y of even length.

15. Prove by induction: If P is a path from a vertex x to a vertex y, then P contains a simple path.

16. Prove that a digraph G is unilaterally connected iff there is a directed path in G containing all the vertices of G.

17. Prove that a digraph G is strongly connected iff there is a closed directed path containing every vertex in G. (A path (v_0,v_1), $(v_1,v_2),\ldots,(v_{n-1},v_n)$ is *closed* if $v_0 = v_n$.)

§ 3.6

18. Solve the following recurrence relations by making an appropriate substitution to transform the relations into linear recurrences with constant coefficients.
    (a) $\sqrt{a_n} - \sqrt{a_{n-1}} - 2\sqrt{a_{n-2}} = 0$ where $a_0 = a_1 = 1$.
    (b) $na_n + na_{n-1} - a_{n-1} = 2^n$ where $a_0 = 10$.
    (c) $a_n^3 - 2a_{n-1} = 0$ where $a_0 = 8$. Hint: let $b_n = \log_2 a_n$.
    (d) $a_n - na_{n-1} = n!$ for $n \geq 1$ where $a_0 = 2$.
    (e) $a_n = \dfrac{\sqrt{a_{n-1}}}{a_{n-2}^2}$ where $a_0 = 1$ and $a_1 = 2$.
    (f) $a_n + 5na_{n-1} + 6n(n-1)a_{n-2} = 0$ where $a_0 = 6$ and $a_1 = 17$.
    (g) $a_n = (a_{n-1})^2\,(a_{n-2})^3$ where $a_0 = 4$ and $a_1 = 4$.
    (h) $na_n = (n-2)a_{n-1} + 2n$ where $a_0 = 5$.
    (i) $na_n = (n+1)a_{n-1} + 2n$ where $a_0 = 1$.

24. Solve the divide-and-conquer relations using a change of variables.
    (a) $a_n = 5a_{n/2} - 4$ where $a_1 = 0$ and $n = 2^k$ for $k \geq 0$.
    (b) $a_n = 2a_{n/3} + 4$ where $a_1 = 5$ and $n = 3^k$ for $k \geq 0$.
    (c) $a_n = 3a_{n/8} + 2n$ where $a_1 = 1$ and $n = 8^k$ for $k \geq 0$.
    (d) $a_n = 5a_{n/3} + n$ where $a_1 = 5/2$ and $n = 3^k$ for $k \geq 0$.

11. Prove by mathematical induction that if R is a relation on A, then
    (a) $R^m \cdot R^n = R^{m+n}$
    (b) $(R^m)^n = R^{mn}$
12. Give an example of a relation R and a positive k such that $R^{k+1} \cdot R^{-1} \neq R^k$.
13. (a) Show that the transitive closure of a symmetric relation is symmetric.
    (b) Is the transitive closure of an antisymmetric relation always antisymmetric?
    (c) Show that the transitive closure of a reflexive and symmetric relation is an equivalence relation.
14. Let R_1 and R_2 be arbitrary binary relations on a set A. Prove or disprove the following assertions.
    (a) If R_1 and R_2 are reflexive, then $R_1 \cdot R_2$ is reflexive.
    (b) If R_1 and R_2 are irreflexive, then $R_1 \cdot R_2$ is irreflexive.
    (c) If R_1 and R_2 are symmetric, then $R_1 \cdot R_2$ is symmetric.
    (d) If R_1 and R_2 are antisymmetric, then $R_1 \cdot R_2$ is antisymmetric.
    (e) If R_1 and R_2 are transitive, then $R_1 \cdot R_2$ is transitive.
15. Let R be a binary relation on a set A where A has n elements. Prove that the transitive closure of $R = \bigcup_{i=1}^{n} R^i$.
16. Prove that if R is a transitive relation on a set A, then for each positive integer n, $R^n \subseteq R$.
17. Let R be a relation on a set A. Prove:
    (a) If R is reflexive, then $R \subseteq R^2$.
    (b) R is transitive iff $R^2 \subseteq R$.
18. Suppose that R and S are relations on a set A, where $R \subseteq S$ and S is transitive. Prove that $R^n \subseteq S$ for each positive integer n.
19. Suppose that R and S are symmetric relations on a set A. Prove:
    (a) If $(x,y) \in S \cdot R$, then $(y,x) \in R \cdot S$.
    (b) If $R \cdot S \subseteq S \cdot R$, then $R \cdot S = S \cdot R$.
    (c) $R \cdot S$ is symmetric iff $R \cdot S = S \cdot R$.
    (d) R^n is symmetric for each positive integer n.
20. Suppose R and S are relations on a set A. Prove or disprove:
    (a) If R and S are reflexive, then so is $R \cdot S$.
    (b) If R and S are both reflexive and symmetric, then $R \cdot S$ is reflexive and symmetric iff $R \cdot S = S \cdot R$.
    (c) If R and S are transitive, then $R \cdot S$ is transitive iff $R \cdot S = S \cdot R$.
    (d) If R and S are equivalence relations on A, then $R \cdot S$ is an equivalence relation iff $R \cdot S = S \cdot R$.
21. Suppose R and S are relations on a set A.
    (a) Prove that $(R \cdot S)^{-1} = S^{-1} \cdot R^{-1}$.
    (b) Is it true that $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$?
    (c) Is is true that $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$?
    (d) If R is an equivalence relation on A, is it true that R^{-1} is an equivalence relation on A?
    (e) If R satisfies any of the six properties of relations defined in Section 4.2, determine whether or not R^{-1} satisfies the same properties.
    (f) Suppose that $R \subseteq S$. Show that $R^{-1} \subseteq S^{-1}$.
22. Assume that R is a reflexive relation on a set A.
    (a) Show that $R \cdot R^{-1}$ is reflexive and symmetric.
    (b) Prove or disprove $R \cdot R^{-1}$ is transitive.
    (c) Prove or disprove that the transitive closure $R \cdot R^{-1}$ is an equivalence relation.
23. Let R and S be relations from A to B and let T and W be relations from B to C. Prove:
    (a) $R \cdot (T \cup W) = (R \cdot T) \cup (R \cdot W)$
    (b) $R \cdot (T \cap W) = (R \cdot T) \cap (R \cdot W)$
    (c) If $R \subseteq S$, then $R \cdot T \subseteq S \cdot T$.

nd Search and

aphs as models
Flow and Min-

Multigraphs
and Four-Color