RELATIONS

Definition of a Relation *1: A relation ~~of~~ on a set X is just a collection of ordered pairs of X.

(An ordered pair is written (a,b) (a & b are elements of the set in question). Note that (a,b) and (b,a) are different ordered pairs. That is $(a,b)=(c,d)$ exactly when $a=c$ and $b=d$.)

Definition of a Relation *2: A relation is a multi-digraph with the ~~property~~ that there is at most one directed edge from any pair of vertices.

(This means there could be one edge from a to b and an edge from b to a there also could be a loop from a to itself.)

These two definitions are the same*. The set X in 1 coresponds to the vertices of the graph in 2. The ordered pair (a,b) in 1 coresponds to the directed edge from a to b. (Note that (b,a) goes the opposite direction.). (Also (a,a) coresponds to a loop from a to a.)

Notation: Sometimes in Definition 1 the relation is denoted by a capitial letter $(a,b)\in R$, aRb are both used at times to say that a is R-related to b or equivalently in Definition 2 that there is a directed edge from a to b.

Examples all of the following are relations

1. $=$ (equal) on any set
2. $<$ (less than), $>$, $\leq$, $\geq$, $\neq$ on any subset of reals
3. "is a child of", "is a parent of" on trees or people
4. "is redder than", "is bigger than", "is unrelated to" (people)
5. "is a subset of", "is implied by", "is connected to"
6. "are both child of the same parents", "have a parent in common"
7. If $X=\{1,2,3,4\}$ $R=\{(1,1),(1,2),(1,3),(3,1)\}$ is a relation

* Actually sets can be infinite and graph have only a finite number of vertices — but we will allow infinite graphs in this chapter.

4. Properties a relation may have:

A. Reflexive: A relation is reflexive if $\forall a \; aRa$
that is in the graph model all the loops are edges.

B. Symmetric: A relation is symmetric if
$\forall a \forall b \quad aRb \Longrightarrow bRa$
in the graph model this means if there is an edge from a to b then there is an edge from b to a.
[note that this doesn't require any edge to be there, just that the non-loops come in pairs.]

C. Transitive: A relation is transitive if
$\forall a \forall b \forall c$ if aRb and bRc then aRc
in the graph model this means if there is an edge from a to b and an edge from b to c then there is an edge from a to c.

D. Anti-symmetric: A relation is anti-symmetric if
$\forall a \forall b \quad aRb \;\&\; bRa \Longrightarrow a = b$
in the graph model it means if there is an edge from a to b and $a \neq b$ (the edge is not a loop) then there is no edge from b to a.

Examples

Relation	Reflexive	Symmetric	Transitive	Anti-symmetric
1 equals	yes	yes	yes	yes
2 less than	no	~~yes~~ no	yes	yes
3 less than or equal to	yes	no	yes	yes
4. is a child of	no	no	no	yes
5. have a parent in common	yes	yes	no	no
6. have both parents in common	yes	yes	yes	no
7. $x - y = 1$ or -1	no	yes	no	no
8. $X = \{1,2,3,4\}$ $R = \{(1,1),(1,2),(1,3),(3,1)\}$	no	no	no	no

3 special kinds of relations:

A. Equivalence Relation: A relation which is reflexive, symmetric and transitive.

(1 & 6 are equivalence relations in the last example)

others include similar or congruent (figures), isomorphic (graphs), give the same remainder when divided by 13.

B. Partial Ordering: A relation which is reflexive, anti-symmetric and transitive

(1, 3 are partial orderings in the last example)

others include "is a subset of" "is an ancestor of" (provided you made everyone his/her own ancestor)

C. Total Ordering: A partial ordering which either has an edge from a to b or b to a.

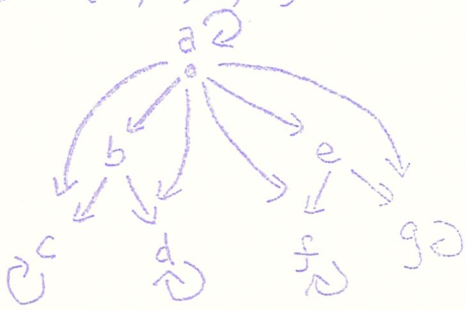
"is a subset of" "is an ancestor of" are <u>NOT</u> total orderings the others in B are.

<u>Exercises:</u> Which of the above ~~examples~~ 7 properties do the following relations enjoy?

1. 

2. 

3. $X = \{1,2,3,4\}$ $R = \{(1,1),(1,2),(2,3),(1,3),(2,2),(3,3),(4,4)\}$

4. 

5. X = real numbers $xRy \iff y \leq x+1$

6. "Has a grandparent in common"

7. X = complex numbers $a+bi \leq c+di \iff a \leq c \;\&\; b \leq d$

8. X = complex numbers $a+bi \leq c+di \iff$ $(a<c)$ or $(a=c$ and $b \leq d)$

9. 

10. " is to the left of" on $\underbrace{\text{books lined up on a shelf}}$

11. " is next to" ditto

12. " is at least as hot as" on stars (like the sun, not people)

MORE EXERCISES:

13. A relation is anti-transitive if aRb and bRc imply that it is not true that aRc.

A. What does this mean in our graph model?

B. Can the graph have any loops? (if it is anti-transitive)

14. A relation is anti-reflexive if aRa is never true.

A. Show that a ordinary graph ~~is~~ canbe thought of as an anti-reflexive symmetric relation. [thinking of the edges as two way paths]

B. Is the reverse also true?

15 If G is any ordinary graph define the following relations on its vertices.

A. " is connected by a path to"

B. " is in the same component as"

C. " is in the same bi-connected component as"

D. " is adjecent to"

E. " has the same colors (in any $\chi(G)$-coloring)"

which of the 7 ~~prop~~ properties these relations have of course depend on the graph G. Which properties to these relations have for all graphs G? Which are never true for any graph G?

16. Two of the eight possible yes & no combinations for reflexive, symmetric and transitive properties are missing from the examples on Pg 2. Find relations with these combinations of yes and no.

17. A function can also be considered a relation. Indeed define $aFb \iff f(a) = b$. Since a function is single valued aFb & $aFc \implies b = c$. What does this mean in our graph model?

18. The transitive closure of a relation R is the relation S where $aSb \iff$ the are $c_1, c_2, \ldots c_n$ so that either aRb or (aRc_1 and c_1Rc_2 and ... and c_nRb)

A. Show S is transitive.

B. What does this look like for our graph model?