MAD 3104 — Discrete Math 1

Section 3, Fall 1995. MW 2:30–3:45. 107 LOVE

Instructor: Bellenot.

The good doctor's Office: 002-B Love, Office Hours: MW 1:25–2:15 or by appointment. Email addressed bellenot@cs.fsu.edu, bellenot@math.fsu.edu, or even bellenot@fsu.edu will get to the good doctor, but the short address 'bellenot' works on math or cs machines.

Eligibility: A grade of C- or better in Pre-Calculus Mathematics (MAC 1140).

Text: Dossey, Otto, Spence and Eynden, *Discrete Mathematics* 2^{nd} Edition.

Coverage: The Appendix, Chapters 2-4 and part of 7 and additional material as time allows.

Final: At 12:30 – 2:30 am Friday, Dec 15, 1995. (Note the late final time.)

Tests: (3) Tentatively at Sept 20, Oct 25?(or Nov 1?) and Nov 29. No Makeup tests.

Quizes: Every Wednesday (except test days). No Makeup quizes.

Grades: 90% A, 80%B, 70%C, 60%D.

Relative Weights F = 2T and T = Q (F is 1/3, each T is 1/6 and Q is 1/6).

Homework and Attendance are required. Indeed attendance will be taken by checking off homework. It is the student's responsibility to see that homework is delivered on time. (The homework needs to be turned in even when the student is absent.) Likewise, being absence is not a valid reason for not knowing the next assignment.

Three or more late or missing homeworks is an automatic FAIL.

Fair Warning: Usually I give the warning below here. However, with the new text, this is not as true as it was under the old text. *The good doctor teaches this course differently than other instructors. The order of material will be different and so will be the emphasis.*

Proofs: At the request of the computer science faculty this class contains lots of proofs. Part of the grade of a proof will be based on the proof having the correct form (in addition to its being a proof or not).

The Web page for the class is "http://www.math.fsu.edu/~bellenot/class.html".

Valid or Not?

1. If today is David's birthday, then today is January 24.
   Today is January 24.
   Hence, today is David's birthday.

2. If the client is guilty, then he was at the scene of the crime.
   The client was not at the scene of the crime.
   Hence, the client is not guilty.

3. The days are becoming longer.
   The nights are becoming shorter if the days are becoming longer.
   Hence, the nights are becoming shorter.

4. If angle $\alpha =$ andgle β, then the lines AB and BC are equal.
   We know $AB = BC$.
   Hence, angle $\alpha =$ angle β.

5. The earth is spherical implies that the moon is spherical.
   The earth is not spherical.
   Hence, the moon is not spherical.

6. If David passes the final exam, then he will pass the course.
   David will pass the course.
   Hence, he will pass the final exam.

7. If the patient has a virus, he must have a temperature above $99°$.
   The patient's temperature is not above $99°$.
   Hence, the patient does not have a virus.

8. If diamonds are not expensive, then gold is selling cheaply.
   Gold is not selling cheaply.
   Hence, diamonds are expensive.

9. AB is parallel to EF or CD is parallel to EF.
   AB is parallel to EF.
   Hence, CD is not parallel to EF.

10. AB is parallel to EF or CD is parallel to EF.
    CD is not parallel to EF.
    Hence, AB is parallel to EF.

Proofs about graphs which are almost trees.

The following statements are equivalent (for a simple graph G):

a. G is a tree (For vertices x and y of G there is a unique xy-path in G).
b. G is connected and acyclic. (Acyclic means it has no cycles.)
c. G is connected and $|E| = |V| - 1$.
d. G is acyclic and $|E| = |V| - 1$.
e. G is connected and each edge e is a bridge. (e is a bridge or cut edge means $G - e$ is disconnected.) [This says a tree is a minimal connected graph.]
f. G is acyclic and if $x, y \in V(G)$ and $e = xy \notin E(G)$ then $G+e$ has a cycle. [This says a tree is a maximal acyclic graph.]

1. Show the following statements are equivalent (G a simple graph):

a. G is connected and has exactly one cycle.
b. G is connected and $|E| = |V|$.
c. G has an edge e so that $G - e$ is connected and acyclic.
d. There is a tree T and an edge $e \notin E(T)$ so that $G = T + e$.

The a, b, c and d above are also equvalent to e below (but you are not asked to prove this.)

e. G is connected, but not a tree, and for vertices x and y of G there are at most two simple xy-paths in G.

2. Show the following statements are equivalent (G a simple graph):

a. There is an edge $e \notin E(G)$ so that $G + e$ is a tree.
b. G is acylic and $|E| = |V| - 2$.
c. G is acylic, disconnected and there is an edge so that $G + e$ is connected.
d. G is the disjoint union of two trees. [G has two components both of which are trees.] And each of the items above implies 2e, but not conversely.
e. If G is disconnected and for x, y vertices of G in different components, then for $e = xy$, $G + e$ is connected.

The following are from old tests.

4. Prove G is a forest $\iff$ every edge of G is a cut edge.

5. Prove by induction (on the number of vertices), if T is a tree, then $|V(T)| = |E(T)| + 1$.

6. Prove if G is acyclic and $|E(G)| = |V(G)| - 2$, then there is an edge $e \in E(\bar{G})$ so that $G + e$ is a tree.

7. Using the formula $\sum_{v \in V(G)} \deg v = 2|E(G)|$, or induction, or any other method, prove that a tree with a vertex of degree 3 has at least 3 vertices of degree 1. [*Hint:* the tree must have at least 4 vertices.]

8. Prove if G is connected and $|E(G)| = |V(G)|$, then there is an edge $e \in E(G)$ so that $G - e$ is a tree.

Proofs: Contradiction and Induction

1. Prove by contradiction: A graph with 100 edges and 19 vertices has a vertex of degree at least 11.
2. Prove by contradiction:
   A. Prove a graph with 35 edges and 16 vertices has a vertex of degree at least 5.
   B. Prove a graph with 45 edges and 24 vertices has a vertex of degree at most 3.
3. Prove by contradiction:
   A. Prove a graph with 40 edges and 25 vertices has a vertex of degree at least 4.
   B. Prove a graph with 50 edges and 20 vertices has a vertex of degree at most 5.
4. Prove by contradiction:
   A. Prove a graph with 41 edges and 20 vertices has a vertex of degree at least 5.
   B. Prove a graph with 49 edges and 25 vertices has a vertex of degree at most 3.

5. Prove by induction (on n) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.
6. Prove by induction that $\sum_{i=1}^{n}(2i-1) = n^2$.
7. Prove by induction that $1 + 3 + 5 + \cdots + (2n-1) = n^2$.
8. Prove by induction: For each integer $n \geq 0, 4^n > n^2$.
9. Prove by induction that $n! > 2^n$ for $n \geq 4$.
10. Given $a_0 = 2$, $a_1 = 0$ and $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, prove by induction, for each integer $n \geq 0$, $a_n = 6 \cdot 2^n - 4 \cdot 3^n$.
11. Given $a_0 = 3$, $a_1 = 0$ and $a_{n+1} = 6a_n - 8a_{n-1}$ for $n \geq 1$, prove by induction, for each integer $n \geq 0$, $a_n = 6 \cdot 2^n - 3 \cdot 4^n$.

Relations

Problems: For the given A and R and each of the properties: A. reflexive, B. symmetric, C. anti-symmetric and D. transitive, decide if R has the property. If it has the property then prove it has that property or if it doesn't have the property then give a counterexample to show the property fails. (I.e. Prove or disprove.)

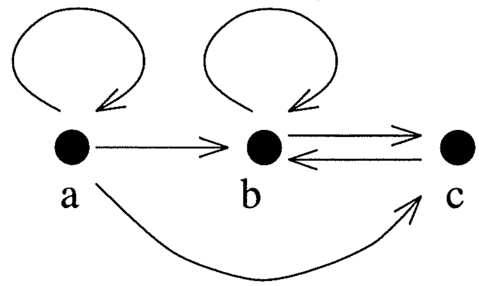
1. A is the set of real numbers and $aRb \iff a \leq b$.
2. A is the set of real numbers and $aRb \iff a < b$.
3. A is the set of real numbers and $aRb \iff 0 \leq a - b \leq 2$.
4. A is the set of real numbers and $aRb \iff |a - b| < 2$.
5. A is the set of odd positive integers and $aRb \iff a \neq b$ and a evenly divides b.
6. A is the set of real numbers and $aRb \iff a^2 - b^2 = 0$.
7. A is the set of positive integers and $aRb \iff a \quad \text{divides} \quad b$.
8. A is the set of integers and $aRb \iff a - b$ is odd.
9. A is the set of positive integers and $aRb \iff a \equiv 1 \mod b$.
10. A is the set of integers and $aRb \iff a \cdot b$ is even.
11. A is the set of points in the plane and $(a,b)R(c,d) \iff (a-c)^2 + (b-d)^2 \leq 5$.
12. A is the set of points in the plane and $(a,b)R(c,d) \iff a + b = c + d$.
13. A is the set of points in the plane and $(a,b)R(c,d) \iff |a - b| = |c - d|$.
14. A is the set of points in the plane and $(a,b)R(c,d) \iff a = c$.
15. A is the set of points in the plane and $(a,b)R(c,d) \iff a = d$.
16. A is the set of triangles in the plane and $tRs \iff$ triangle t has the same area as triangle s.
17. A is the set of triangles in the plane and $tRs \iff$ triangle t is similar to triangle s.
18. A is the set of triangles in the plane and $tRs \iff$ triangle t has either at least as much area as triangle s, or triangle t has at least as large perimeter as triangle s.
19. A is the set $\{1, 2, 3, \{1\}, \{1, 3\}, \{2\}\}$ and $aRb \iff a \in b$.
20. A is the set power set of $\{1, 2, 3\}$ and $aRb \iff a \subseteq b$.

MAD 3104 Discrete Math 1 **Test 1** 20 Sep 1995

Show **ALL** work for credit; be neat; and use only **ONE** side of each page of paper.

1. Use De Morgan's and/or the distributive laws to simplify:
 A. $\overline{A} \cap (A \cup B)$
 B. $\overline{(A - B)} \cap A$

2. Give counterexamples to the statements below. The relation R is the one given by the digraph below.
 A. R is reflexive.
 C. R is symmetric.
 D. R is anti-symmetric.
 E. R is transitive.



3. Solve for m in Z_{13}. Write your answer so that $0 \leq m < 13$.
 A. $[5] + [12] = [m]$
 B. $[5][9] = [m]$
 C. $[5][m] = [3]$
 D. $[7]^{101} = [m]$

4. Construct a truth table for $(p \wedge q) \rightarrow (\sim p \vee q)$.

5.For each part, decide whether the logic is valid or invalid and draw a Venn diagram to support your answer.
 A. If $x + 2 = x$, then x is blue. $x + 2 = x$. Therefore, x is blue.
 B. If T is a rectangle, then T is a square. T is a square. Therefore, T is a rectangle.

6. Negate the following statements and re-write them so that words like "not" or "no" are not used.
 A. For all triangles T, the area(T) $\geq$ perimeter(T).
 B. For some integers x, x is odd and x^2 is even.

7. Draw the digraph fot the relation R on the set $A = \{2, 3, 4, 5, 6\}$ where the relation R is defined by $aRb \iff a = 3$ or $b = 5$.

8. Equivalent classes. For the given set A, the relation R is an equivalence relation, describe the the equivalence class $[p]$, for the given p.
 A. A is the set of reals, $aRb \iff a^2 = b^2$, and $p = 4$.
 B. A is the points in the plane, $(a,b)R(c,d) \iff a^2 + b^2 = c^2 + d^2$, and $p = (3, 4)$.
 C. $A = \{1, 2, 3, 4, 5, 6\}$, $aRb \iff a = b$ or both $a \geq 3$ and $b \geq 3$, and $p = 4$.

9. A is the set of reals and $aRb \iff a + 10 \leq b$.
 A. Give counter-examples to show R is not reflexive and not symmetric.
 B. Give proofs to show R is anti-symmetric and transitive.

10. Proof or disprove. Let A be the set of points in the plane and let R be the relation defined by $(a,b)R(c,d) \iff |a - c| \leq |b - d|$.
 A. R is reflexive.
 B. R is symmetric.
 C. R is anti-symmetric.
 D. R is transitive.

Keep this sheet. Graded tests will be ready 1:30 Friday 22 Sep in 002B LOV.

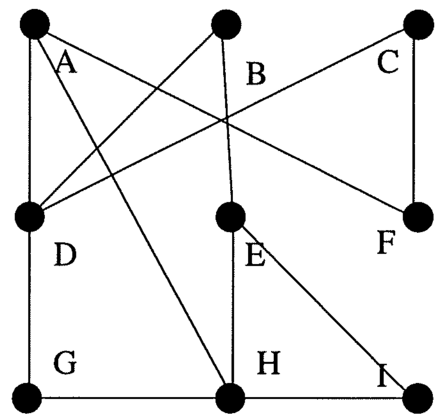
MAD 3104 Discrete Math 1 **Test 2** 25 Oct 1995

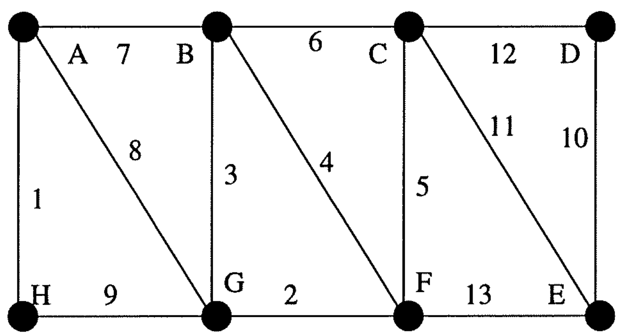
Show **ALL** work for credit; be neat; and use only **ONE** side of each page of paper.

1. Two unrelated short problems.
 A. How many ways can a 333-element subset be selected from a set with 666 elements?
 B. Make a pairwise nonisomorphic list of all 6 trees with 5 edges.

2. For the graph to the right.
 A. Find the DFS spanning tree.
 B. Find the BFS spanning tree.

3. For the graph to the right
 A. Find a cycle.
 B. Find a circuit that is not a cycle.
 C. Find an Euler path.

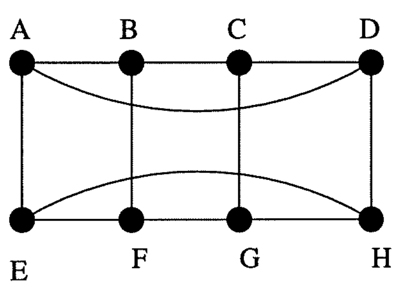


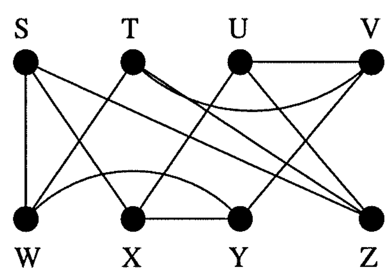


4. For weighted graph above, list the edges IN THE ORDER SELECTED by:
 A. Prim's Algorithm (starting at A).
 B. Kruskal's Algorithm.

5. For the weighted graph above problem #4 apply Dijkstra's Shortest Path Algorithm starting at vertex A. Show the RESULTING labels for each vertex, show the shortest PATH obtained from A to E and STATE its length. (Be sure to answer ALL parts of this question!)

6. Decide if the two graphs below are isomorphic or not. If they are isomorphic, then give an isomorphism. If they are not isomorphic, then explain why they are not isomorphic.





7. Prove by induction (on n) $1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$.

8. Prove by contradiction: Prove a graph with 61 edges and 30 vertices has a vertex of degree at least 5.

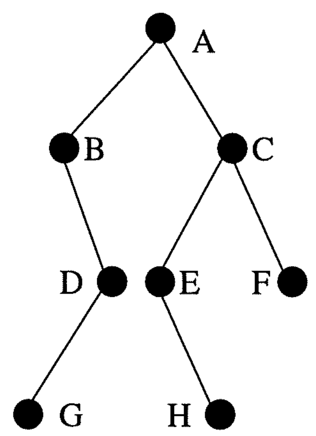
9. Given $a_0 = 0$, $a_1 = 3$ and $a_{n+1} = 2a_n + 7a_{n-1}$ for $n \geq 1$. Prove by induction, for each integer $n \geq 0$, $a_n < 4^n$.

10. Prove by induction: For each integer $n \geq 0$, $\frac{(2n)!}{n!2^n}$ is an integer. [By definition $0! = 1$.]

MAD 3104 Discrete Math 1 **Test 3** 29 Nov 1995

Show **ALL** work for credit; be neat; and use only **ONE** side of each page of paper.

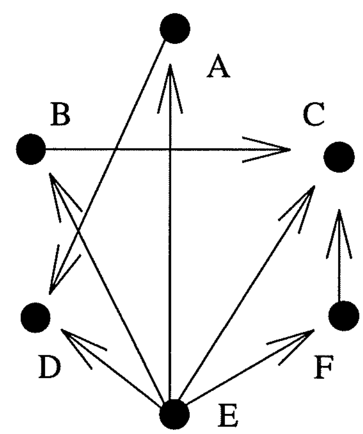
1. For the binary tree to the right, list the vertices in:
   A. Preorder.
   B. Inorder.
   C. Postorder.



2. Draw the binary tree.
   A. BST for the data 60, 10, 80, 20, 50, 70, 40.
   B. For the expression $((a+b)*(r-s)+7)/(u*v-x/y)$.

3. Draw the binary tree.
   A. With level-order vertices 1, 2, 3, 5, 6, 10, 13, 27, 54, 55.
   B. The vertices in postorder are $BAFGDCE$ and the vertices in inorder are $BEFACGD$.

4. For the digraph to the right.
   A. Give the in-degree and out-degree of each vertex.
   B. Write the adjacency matrix for the digraph.



5. Construct the binary tree. (Smaller weights to the left.)
   A. The Optimal binary tree for the weights 10, 12, 13, 16, 17 and 18.
   B. Give the resulting code word for each weight.

6. What is the chromatic number of the following graphs?
   A. K_5 – the complete graph on 5 vertices.
   B. The empty graph (empty of edges) with 3 isolated vertices.
   C. $K_{3,3}$ – the utility graph.
   D. Any non-trivial tree.

7. Give counterexamples. (In C and D be sure to label the edge e.)
   A. Every tree has a vertex of degree 1.
   B. A graph with $|E| = |V| - 1$ is a tree.
   C. In a graph G with $|E| = |V|$ any edge e in G will make $G - e$ a tree.
   D. In a graph G with $|E| = |V| - 2$ any edge e not in G will make $G + e$ a tree.

8. Prove: If e is an edge not in G so that $G + e$ is a tree, then G is acyclic and $|V(G)| - 2 = |E(G)|$

9. Prove: G is connected and $|E| = |V|$ then G has an edge e so that $G - e$ is a tree.

10. Prove by (stong) induction on the number of **CYCLES:** Every connected graph satisfies $|E| \geq |V| - 1$.

MAD 3104 Discrete Math I Section 3: Day by Day

Class Handout

1. M 28 Aug: Homework 1-10 on relations handout. Defined relations, reflexivity, symmetric, anti-symmetric and transitive. Counter- examples and proofs. Venn diagrams. Vacuously true. [hmmm, the text does not define anti-symmetric. Relations and things in 2.2. Venn diagrams in 2.1. The appendix (mostly A.3) talks about proofs.]
2. W 30 Aug: Homework 2.1 1-25 odd?, 11-17 on relations handout., and determine which of the relations on the digraph handout. Defined relations, are reflexive, transitive, symmetric and/or anti-symmetric. Equivalence relations and partial orders, Relations as Di-Graphs, Set Theory and review of the homework. [Set Theory in 2.1, Relations 2.2]
3. W 6 Sep: Homework 2.2. Draw the di-graphs for 1 and 2. Prove or disprove 1-12 and do 13-17. Also read appendix 1 and do problems 1-12. Equivalent relations, partitions and equivalence classes all in 2.2. Quiz 1.
4. M 11 Sep: Homework A.1 13-31 = 1 mod 2; A.2 1-17 = 1 mod 4; 2.3 = 1 mod 6; Negation, Truth Tables, Congruence. It was from the text for a change.
5. W 13 Sep: Homework A.1 33-36, 1-16 on the Valid or Not handout. and the two mod 13 problems: 5 to the 1000, and solving [7][m]=[3]. Valid and Invalid reasoning (with Venn diagrams); Converse, Inverse, Contrapositive. Quiz 2.
6. M 18 Sep: Review day for test 1. Homework due today will be collected next monday. Why do clocks go clockwise?
7. W 20 Sep: Test 1.
8. M 25 Sep: Homework 2.5 12, 13, 18, 19; 3.1 1, 5, 19-21, 48; Introduction to graphs, Induction. Test debriefing.
9. W 27 Sep: Homework 2.5 14, 15, 20, 21; 3.1 26-31, 49, 51, 52. More Induction, more graphs. Quiz 3.
10. M 2 Oct: Homework 2.3 39-42; 2.5 16, 17, 24, 25; 3.2 1-20; All Graphs with $|V| = 5$; All Trees with $|V| = 6$.
11. W 4 Oct: Opal cancels class.
12. M 9 Oct: Homework: More Proofs Handout 2, 6, 7; 4.1 1-17; All Trees with $|V| = 7$.
13. W 11 Oct: Homework: 2.4 65-68; More Proofs Handout 3, 12; 4.2 1-8. BFS Trees, rooted trees, ordered trees. Quiz 4.
14. M 16 Oct: Homework: 2.6 1-31=1 mod 4; 3.2 35, 37, 40; 3.3 1-15odd More proofs handout 4, 10; 4.3 1. Dijkstra's, Prim's
15. W 18 Oct: Homework: 2.6 31,32; 3.2 48-50; 4.3 2-4, 22-25; 4.4 1-6 (find DFS spanning tree as well). Kruskal's, DFS. Quiz 5
16. M 23 Oct: Review for test2; Homework due today will be collected 30 Oct.
17. W 25 Oct: Test 2
18. M 30 Oct: Strong Induction. Homework: Prove by Strong induction on the number of edges, a tree with n edges has n + 1 veritces; 3.2: 53, 54; 3.3 14
19. W 1 Nov: The Almost Tree Proof Page And Quiz 6 Homework: Prove by Strong induction on the number of vertices, a tree with n vertices has n - 1 edges; Almost trees 1a iff 1b; A.3 13, 14.
20. M 6 Nov: Binary Search Trees, Preorder, Postorder, Inorder, Infix, Postfix, Prefix. Homework: Prove by Weak induction on the number of vertices, a tree with n vertices has n - 1 edges; Almost

trees 1a,b,c,d are equivalent; Prove by contradiction: a non-trivial tree with n vertices, n - 1 edges has a vertex of degree at most 1; 4.6 problems 1-25 = 1 mod 6.

21. W 8 Nov: Graph coloring, more proofs. SIRS Homework: Almost trees problem 2; 3.4 1-25 odd; 4.6 47-57 odd, 67; 4.7 48, 51, 52.
22. M 13 Nov: Level order numbering: Homework Almost trees problems 4-6; Find the smallest binary tree containing vertices with level order numbers 11, 103, and 106; Find the left child, level and parent of the vertex with level-order number 135.
23. W 15 Nov: Homework Almost trees #7, 4.7 1-33 odd Quiz 7.
24. M 20 Nov: Homework Almost trees #7, 3.5 1-39 odd
25. W 21 Nov: Homework 4.5 1 - 31? odd Quiz 8.
26. M 27 Nov: Review Homework
27. W 29 Nov: Test 3.
28. M 4 Dec: Homework psuedo code Prim's Algorithm
29. W 6 Dec: Last Day of classes
30. F 15 Dec: 12:30-2:30 Final.

last update 4 Dec 95.