1

Chapter 0 Logic and proofs.

The importance of using correct logic cannot be over emphasized. An arithmetic error will usually make the answer wrong, but changing a few numbers will make it right. Algebraic errors can be more serious, but often they too are easy to correct. In contrast, a logical error will usually ruin everything. It is like puting too much salt on your dinner, generally the best solution is to throw it out and start over again.

The logic here is presented in an informal matter. But the reader is warned that there is always a certain formally about logic. The first two sections of this chapter are about "What ~~to~~ do statements say?" The third is on syllogisms. The fourth contains some general remarks on proving theorems. And finally the fifth contains common errors and fallacies. The examples are drawn ~~for~~ from plane geometry.

Bellenot 27 Aug 82

#2

18 pgs

§1 Picturing statements.

Let's start with an example. Consider quadrilaterals (~~a~~ four-sided polygons in the plane). There are several properties a quadrilateral may or may not have. For examples:

A. Parallelogram: when ~~so~~ the non-adjacent sides are parallel.

B. Rectangle: a parallelogram all of whose angles are right angles.

C. Rhombus: a parallelogram all of whose sides are of equal length.

How are the properties of being a rectangle and being a rhombus related?

We start with the picture or Venn diagram:

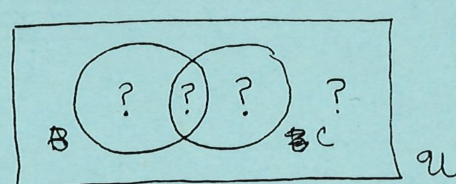


Figure 1.1

(This is a general picture which will fit all two property relations.) Here B is the collection of all rectangles, C is the collection of all rhombi and $\mathcal{U}$ is "the universe" under discussion.

(Usually $\mathcal{U}$ is only implicitly stated by context. Here $\mathcal{U}$ is the set of all ~~q~~ quadrilaterals. (Because of the second sentence in this section.) But a slight change in the wording above would make $\mathcal{U}$ the set of all parallelograms.)

Each of the regions in figure 1.1 has a question mark to indicate that this region could be empty or not. As we gain information, we will change the question mark either to a "$\emptyset$" (void set) to indicate that that region is empty or to a "*" to indicate that that region is non-empty (i.e. there is something inside).

Of course, in this particular example all the regions are non-empty, so obtaining:

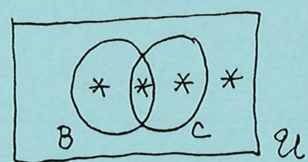
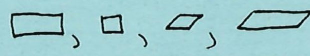


figure 1.2

(Examples ). Although, figure 1.2 has a nice filled-in look, note the picture is still correct if any or all the "*"'s are replaced by "?"'s. Such replacements would result in a loss of information but NOT a loss of truth or validity. By the way, figure 1.2 alone answers our question of the relation between rectangles and rhombi. In a word, there is "none".

Next consider the relationship between rectangles and parallelograms. We obtain figure 1.3, where B is the collection of rectangles and A the collection of parallelograms.

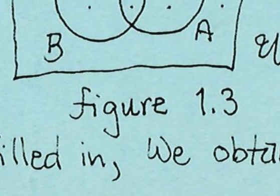


figure 1.3

When figure 1.3 is completely filled in, we obtain figure 1.4 below.

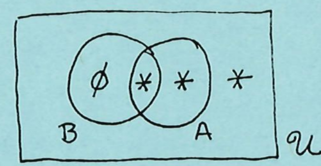


figure 1.4

Figure 1.4 can be re-drawn as figure 1.5 (which is sometimes called Euler circles)

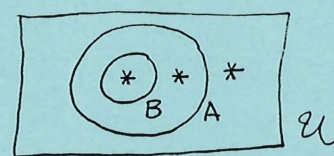


figure 1.5

We can state from either of these last figures several true statements

(1) All rectangles are parallelograms
(2) There are rectangles
(3) There are parallelograms which are not rectangles
(4) There are quadrilaterals which are not rectangles or parallelograms.

Note that these statements correspond to the marks ϕ, *, *, * from left to right in figure 1.4.

Note there is a considerable difference between statements (2), (3) and (4) and statement (1). For the moment we will put ~~up~~ off considering statements like (1), although ~~it is~~ they are perhaps more important for what ~~forms~~ follows.

Statements (2), (3) and (4) are called existence statements. Perhaps they seem obvious, but since we will consider arbitrary properties, there is always the chance that nothing has the property (i.e five-sided quadrilaterals). It is important to note that statements like (2) can be made in many equivalent forms. It is hoped that the following list includes them all.

(2.1) There is at least one rectangle.
(2.2) There are some rectangles.
(2.3) Some things are rectangles.
(2.4) There exist rectangles.
(2.5) rectangles exist.

(There are other ways of saying this but they use the word "not" (see the next section).) (These statements occur so often, that the backwards E ($\exists$) is used ~~to~~ as a symbol for "there exists" thus ~~some~~ another equivalent statement is
(2.6) $\exists$ rectangles)

(Do not be confused by the use of plurals, all these statements just say there is at least one. The statement
(5) There are at least two rectangles.
is not the same as ~~though~~ those above. Statement (5) claims alot more (twice as much).)

Now back to statement (1). In pictures this is equivalent to figure 1.6 below.

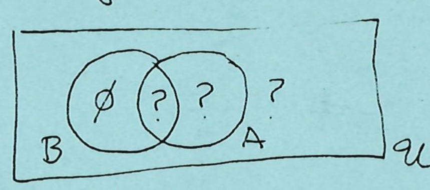


figure 1.6

(Note the "?"'s, statement (1) says nothing about existence of anything. It only states the non-existence of a rectangle which isn't a quadrilateral.) Figure 1.6 represents an <u>implication</u>. Usually these are stated using the "if... then..." or using the "implication symbol" ($\Rightarrow$). Equivalent statements include (many more be...
(1.1) If x is a rectangle, then x is a parallelogram.
(1.2) x a rectangle $\Rightarrow$ x is a parallelogram.

Almost all statements which you will be asked to prove will be implications. It is important to note that there is no causality required for an implication to be true. For instance $A \Rightarrow B$ is true if either nothing is in A or everything is in B. In pictures either of

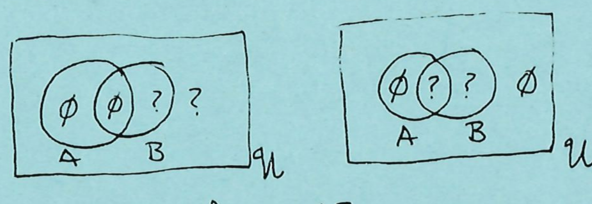


figure 1.7

contain figure 1.6 and so $A \Rightarrow B$ is true. (Examples "If 2+2=5 then it rains every weekend in tallahassee", and "If x is a rectangle then x=x" are both true.)

It is perhaps curious that an implication generally tells you how to loose information. Saying that a rectangle is a quadrilateral tells you less about the the figure than you already know. Perhaps paradoxically, this can be a real advantage (ridding of unnecessary details). Of couse, this simplification may go to far. (On the other hand Many problems become simpler after making them more complicated first.)

The used of the word "all" in statement (1) (and its implicit ~~were~~ use in (1.1) and (1.2)) provides more equivalent statements:

(1.3) each rectangle is a parallelogram.

(1.4) any rectangle is a parallelogram.

(1.5) a rectangle is a parallelogram.

(1.6) every rectangle is a parallelogram.

The words "all", "each", "any" and "every" say the same thing, namely "without exception".

It is important to see (1.5) among the others. To prove something about "all rectangles" (a "universal statement" in contrast to the "existence statements" above.) It suffices to prove it for an "arbitrary" rectangle. (One must be careful not to lose the "arbitrariness" of your rectangle by ~~doing~~ using something that all rectangles don't have.)

(Universal statements have a symbol too, it is an upside down A ($\forall$) (read " for all"). ~~;~~ Thus yet another equivalent statement is

(1.7) $\forall x$ (x is rectangle $\Rightarrow$ x is a parallelogram)).

Finally for completeness, ~~ts~~ two properties or statements A and B are equivalent if they satisfy fiquire 1.8 or (equivalently) $A \Rightarrow B$ and $B \Rightarrow A$.

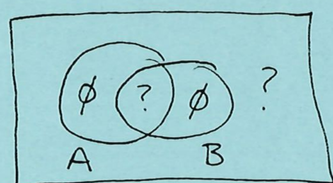


figure 1.8

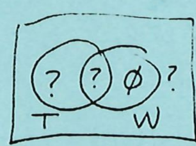
For example equilateral triangles and equiangluar triangles are equivalent. (Note equilateral quadrilaterals are rhombi and equiangluar quadrilaterals are rectangles.)

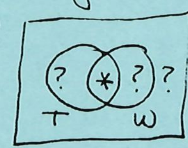
EXAMPLE OF EXERCISES:

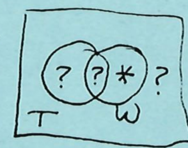
Draw a Venn diagram for each of the following statements. Which implies the ~~tog~~ the other and which are equivalent.

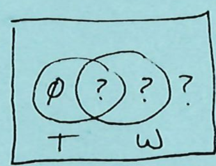
1. All whatits are thingies
2. Some thingies are whatits
3. Not all whatits are thingies
4. All thingies are whatits
5. ~~Nothing~~ No~~t~~ whatits is not a thingies
6. Some whatits are thingies

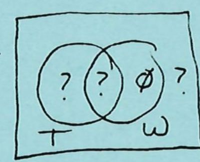
Solutions: Let W be the collection of whatits and T the collection of thingies.

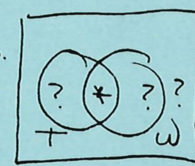
1


2


3.


4.


5.


6.


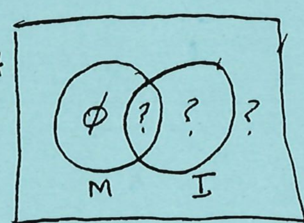
1 and 5 are equivalent.

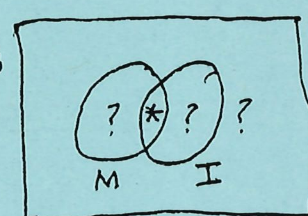
2 and 6 are equivalent.

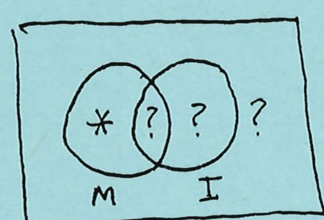
There are no implications other than these equivalencies

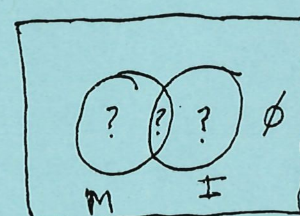
Exercises. Draw a Venn diagram for each. Which are equivalent? Which implies which?

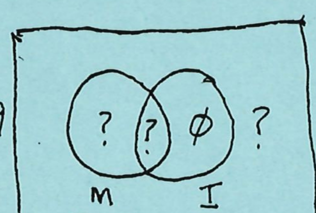
Solutions:

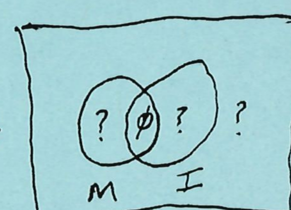
1,3,4


7,12,13


2


10,11


5,6,8,9


14


There are no implications other then these equivalences

§2 Compound Statements.

Let "A" and "B" be statements. We can form the compound statements "not A", "A and B", "A or B", "$A \Rightarrow B$" and "$A \Leftrightarrow B$". Truth tables can be used to determened the validity of these statements given the validity of A and B. The letters T and F represent true and false respectively

A	not A
T	F
F	T

A	B	A and B	A or B	$A \Rightarrow B$	$A \Leftrightarrow B$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Observe that "or" is the inclusive or meaning either A or B or both. The perhaps strange definition of "$\Rightarrow$" is partly convention and it is best explained by the last section. The symbol "$\Leftrightarrow$" is called "equivalent" or "if and only if"

Example:

A is true and B, C and D are false. What about

not $(A \Rightarrow(((B \Rightarrow C)$ and $D)$ or $(A \Leftrightarrow (C \Rightarrow D))))$?

Solution:

$B \Rightarrow C$, $C \Rightarrow D$ are Tree; $(B \Rightarrow C)$ and D is false, $A \Leftrightarrow (C \Rightarrow D)$ is true. Thus the "or" statement is true and the $A \Rightarrow$ "or statement. Thus the "not" makes the whole thing false.

<u>Example</u>: Show $(A \Leftrightarrow B) \Longleftrightarrow ((A \Rightarrow B) \text{ and } (B \Rightarrow A))$ is always true no matter what A & B are.

Solution:

We check all possibilies. Since the last column is all T's it is always true.

A	B	$A \Leftrightarrow B$	$A \Rightarrow B$	$B \Rightarrow A$	$(A \Rightarrow B)$ and $(B \Rightarrow A)$	whole thing
T	T	T	T	T	T	T
T	F	F	F	T	F	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T

The above is called propositional calculus. And it is very limited if we require the statements to be either true or false. Things are much more interestin if we allow statements like "x is a rectangle" which are sometimes true and sometimes false, For these kind of statements, the compound statements a true if there is no x for which they are false. (ie $A \Rightarrow B$ if the case A true but B false does not occur for each x i.e we have the picture

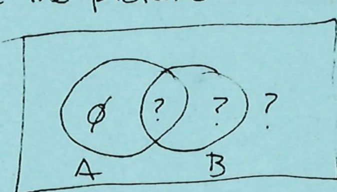


figure 2.1

which is the same as before. Of course some compound statements will be true some of the time and false other times.

The statements A or B may contain the "quantifiers" ~~"qualifiers"~~ $\forall$, $\exists$. For example $[\forall x\ (x \text{ is a rectangle} \Rightarrow x \text{ is a parallelogram})] \Leftrightarrow [\forall x\ (x \text{ is a parallelogram or } \{\text{not } (x \text{ is a rectangle})\}]$ is true. The statement $\forall x\ A$ (respectively $\exists x\ A$) is true if for each x, A is true (respectively, for at least one x, A is true). Usually this is interesting only if the statement A contains x. For this reason, these statements are written $\forall x\ A(x)$ $(\exists x\ A(x))$. The notation $A(x)$ indicates the statement A might depend on x.

Perhaps the most confusing to the beginer are the rules:

(1) "not $(\forall x\ A(x))$" is equivalent to "$\exists x$ not $A(x)$"

(2) "not $(\exists x\ A(x))$" is equivalent to "$\forall x$ not $A(x)$"

However, they are quite logical. If $\forall x\ A(x)$ is false there is an exception. And for (2), if there is no exception then it is always true.

Example:

Move the "not" through the quantifiers

not $(\forall x\ \exists y\ \forall z\ \forall w\ \exists t\quad A(x,y,z,w,t))$.

Solution:

$\exists x\ \forall y\ \exists z\ \exists w\ \forall t$ not $A(x,y,z,w,t)$.

The second most confusing point is the order dependency of the quantifiers. $\forall x\ \exists y$ and $\exists y\ \forall x$ are very different. Consider

(3) For every girl there is a boy

(4) ~~For~~ There is a boy for every girl

Statement (3) says there is a mate for each girl whereas Statement (4) says there is one boy that can satisfy every girl (which is clearly untrue, but (3) is likely to be true.)

We the use of "not" we can form several new statements equivalent to statements (1) and (2) in the last section. In fact infinitely many. A few examples.

(1.8) there is no rectangle which is not a parallelogra

(1.9) not not ($\forall x$ [x is a rectangle $\Rightarrow$ x is a parallelogram]

(1.10) not $\exists x$ not (x is a rectangle $\Rightarrow$ x is a parallelog

⋮

(2.7) not not $\exists$ rectangles

(2.8) not $\forall x$ (x is not a rectangle)

(2.9) $\exists x$ not (x is not a rectangle)

⋮

Note that "not not" is the same as not being there at all.

§3 Syllogisms

A Syllogism~~s~~ is a collection of statements of the form:

All men are mortal
Socraties is a man
∴ Socraties is Mortal

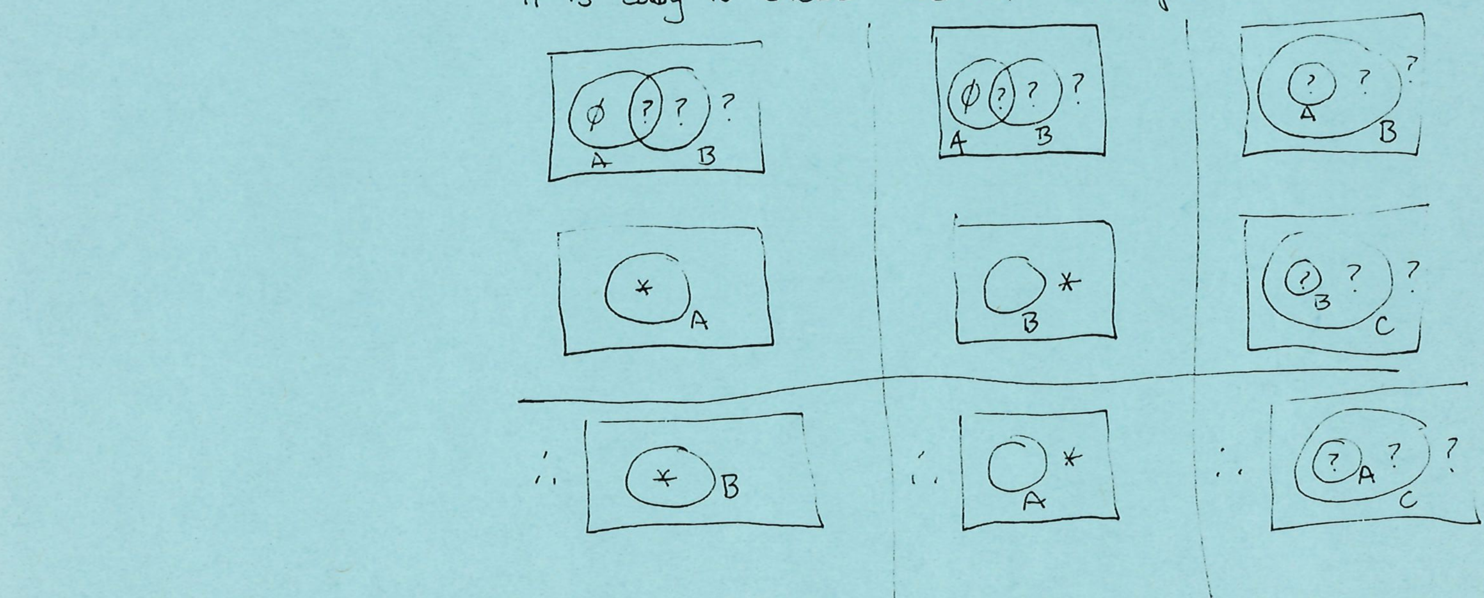
or more generally

$$\begin{array}{c} A \text{ is } B \\ \underline{\quad A \quad} \\ \therefore B \end{array}$$

There are a couple of variations

$$\begin{array}{c} A \text{ is } B \\ \underline{\text{not } B} \\ \therefore \text{not } A \end{array} \qquad \begin{array}{c} A \text{ is } B \\ \underline{B \text{ is } C} \\ \therefore A \text{ is } C \end{array}$$

These are valid reasoning. Unfortunately there are many more example of non-valid reasoning. Fortunateiy it is easy to check these via the diagrams in §1.



(Notice the seemingly lost of information.)

Examples: Which are valid

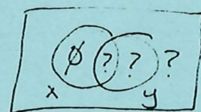
1. each x is a y
   w is a y
   ___
   ∴ w is a x

2. each x is a y
   w is not an x
   ___
   ∴ w is not a y

3. each triangle is a square
   each square is a circle
   ___
   ∴ each triangle is a circle

4. no cat has nine tails
   each cat has one more tail than no cat
   ___
   ∴ each cat has ten tails

Solutions:

For 1 & 2 we have  the second statement gives ~~the~~ the existence of an element but we can't tell which of the regions it belongs. So they are invalid.

3. is valid (even though all the statements are false)

4. is invalid (no cat is used with two different meanings)

Solutions to exercises

The valid ones are: 1, 4, 7, 9, 12, 13, 16, 17, 18, 20, 26, 28, 30
(14 is valid but not a syllogism)

§4 Proofs

Most proofs are (or can be broken down to statements) of the form $A \Rightarrow B$. There are essentially three strategies that might work.

(1) The direct proof.
Assume A and reason logically to B

(2) The indirect proof
Assume not B and reason logically to not A

(3) Proof by contradiction

Assume A and not B and reason logically to a contradiction (a statement that can't be true).

Statements like "$A \Leftrightarrow B$" or "A if and only B" are broken into two parts: "$A \Rightarrow B$" and "$B \Rightarrow A$". To prove $A \Rightarrow (B \text{ and } C)$, prove "$A \Rightarrow B$" and "$A \Rightarrow C$". To prove $A \Rightarrow (B \text{ or } C)$, prove $(A \text{ and not } B) \Rightarrow C$ (or reverse B & C). To prove "$A \Rightarrow B$" is false, give a counterexample — something that has A but not B.

Often many statements are collected together in T.F.A.E. statements (The Following Are Equivalent). For example
T.F.A.E.:

A

B

C

Means $A \Leftrightarrow B$, $B \Leftrightarrow C$ and $A \Leftrightarrow C$. BUT you do not have to prove all six directions. It suffices to show $A \Rightarrow B$, $B \Rightarrow C$ and $C \Rightarrow A$ (Why?). (An alternate way would be $A \Rightarrow C$, $C \Rightarrow B$, $B \Rightarrow A$.)

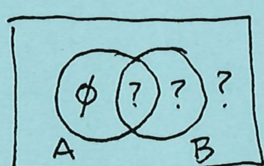
Here is a good place to introduce some standard definitions. From the implication statement $A \Rightarrow B$ we can form three other statements that have names.

The <u>converse</u> of "$A \Rightarrow B$" is "$B \Rightarrow A$"

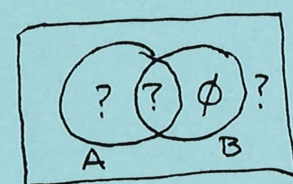
The <u>inverse</u> of "$A \Rightarrow B$" is "(not A) $\Rightarrow$ (not B)"

The <u>contrapositive</u> of "$A \Rightarrow B$" is "(not B) $\Rightarrow$ (not A)"

Looking at the Venn diagrams for these four statements are



$A \Rightarrow B$
& its contrapositive
not B $\Rightarrow$ not A



its converse $B \Rightarrow A$
& its inverse not A $\Rightarrow$ not B

Note that the converse & inverse say something quite different from the original statement. Also the converse and inverse are contrapositives of each other. The original statement is the contrapositive of the contrapositive. Since the contrapositive is equivalent to the original statement, we have the choice of proving it instead of the origin. Indeed this is exactly "the indirect proof" on the preceding page.

~~Finally, we close this section~~

When proving $A \Rightarrow B$ keep in mind that it must be true for all possible cases, so be sure to handle the wrost possible. On the otherhand when proving $\exists A$, you only need one so start with the nicest possible.

MATHEMATICAL INDUCTION: The most important technique of proof for computer science is induction! An induction proof has two parts, "a start up" and "an induction step" There are several forms these can take, but the induction step will determine what you must do for the start up.

The classic induction step is "if it is true for n, then it is true for $n+1$" (It is important that the integer is increasing) In this case the start up is just proving it is true for $n=1$.

But if your induction step is "if it is true for n and true for $n+1$, then it is true for $n+2$" then the start up must include both $n=1$ and $n=2$. (Otherwise there would be no way of knowing if it is true for $n=2$.)

The most common error in induction proofs is proving the converse of the induction step. Namely "if it is true for $n+1$, then it is true for n." Of course, this is useless.

§5 Errors and fallacies

The below are false "proofs" with the most common errors. (Note that they have names.)

Proof By Example:

Thm: All odd numbers are prime

pf: Let x be an arbitrary odd number say 3. 3 is prime.

Note: 9 is odd but not prime

Begging the question:

Thm: All odd numbers are prime

Pf: Let x be an odd number. If x is not prime then x is not odd $\therefore$ odd numbers are prime.

Note: The second statement is a restatement of what you are trying to prove. Just spinning your wheels.

Asserting the converse

Thm A odd numbers greater than 2 are prime

pf: Let x be a prime greater than 2. x must be odd. Otherwise 2 would divide x.

Note: What is proved is the converse. If x is prime greater than 2 than x is odd. Not the same thing.

Proof by intimidation:

pf: Because I say so.

Proof by faith

pf: This is obvious

Proof by Confusion

Thm: $A \Rightarrow B$

pf: Assume A. "The text of E.T." $\therefore$ B

Recurrence Relation Problems

1. $a_n - 2a_{n-1} = 0$ $\quad a_0 = 3$
2. $a_n = 6a_{n-1} + 3$ $\quad a_0 = -8/5$
3. $a_n + 3a_{n-1} = n+2$ $\quad a_0 = 37/16$
4. $2a_n - a_{n-1} - 2^n = 0$ $\quad a_0 = 1$
5. $3a_n = -2a_{n-1} + 3n4^n$ $\quad a_0 = 2$
6. $a_n - 6n - 10 = a_{n-1}$ $\quad a_0 = -3$
7. $a_n = 3a_{n-1} + 2 \cdot 3^n$ $\quad a_0 = 10$
8. $a_n - 2a_{n-1} = -4n2^n$ $\quad a_0 = 5$
9. $a_n + 4a_{n-1} = 10n^2 + 1$ $\quad a_0 = 1$
10. $a_n + a_{n-1} - 4n(-1)^n = 0$ $\quad a_0 = -3$

11. $a_n - 5a_{n-1} + 6a_{n-2} = 0$ $\quad a_0 = 5 \quad a_1 = 12$
12. $a_n = 5a_{n-1} - 4a_{n-2} + 28n$ $\quad a_0 = 0 \quad a_1 = -\frac{136}{7}$
13. $a_n + a_{n-1} = 6a_{n-2}$ $\quad a_0 = 10 \quad a_1 = 0$
14. $a_n = a_{n-2} + 2^n$ $\quad a_0 = 1 \quad a_1 = 1$
15. $a_n - 72n^2 + 5a_{n-1} + 6_{n-2} = 0$ $\quad a_0 = \frac{175}{12} \quad a_1 = \frac{451}{12}$
16. $2a_n + 3a_{n-2} = 7a_{n-1} + 9n2^n$ $\quad a_0 = -32 \quad a_1 = -90\frac{1}{2}$
17. $6a_n + a_{n-1} = 2a_{n-2} + 5n + 13$ $\quad a_0 = 3 \quad a_1 = \frac{35}{6}$
18. $a_n + 4a_{n-1} - 45 + a_{n-2} = 0$ $\quad a_0 = 10 \quad a_1 = -3$
19. $a_n + 9a_{n-2} = 6a_{n-1} + 8$ $\quad a_0 = 2 \quad a_1 = 20$
20. $a_n = -a_{n-2}$ $\quad a_0 = 2 \quad a_1 = 0$
21. $a_n = 2a_{n-1} + a_{n-2} + 4(-1)^n$ $\quad a_0 = 4 \quad a_1 = -4$
22. $a_n + 2a_{n-2} = 2a_{n-1} + 25n3^n$ $\quad a_0 = 22 \quad a_1 = 67$
23. $a_n - 8a_{n-1} + 15a_{n-2} = 12 \cdot 3^n$ $\quad a_0 = 0 \quad a_1 = -8$
24. $a_n + 5 + 2a_{n-2} = 3a_{n-1} + 2n$ $\quad a_0 = -1 \quad a_1 = -1$
25. $a_n - 2a_{n-1} + a_{n-2} = 6n - 10$ $\quad a_0 = 2 \quad a_1 = 2$
26. $a_n = 4a_{n-1} - 4a_{n-2} + 6 \cdot 2^n$ $\quad a_0 = 5 \quad a_1 = 16$

27. $a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3} = 88n - 30$ $\quad a_0 = 1 \quad a_1 = 3 \quad a_2 = 25$
28. $a_n - a_{n-4} = 0$ $\quad a_0 = 4 \quad a_1 = 2 \quad a_2 = 4 \quad a_3 = 2$
29. $a_n - 5a_{n-1} + 8a_{n-2} - 4a_{n-3} = 2^n$ $\quad a_0 = 0 \quad a_1 = 0 \quad a_2 = 8$
30. $a_n - 2a_{n-2} + 4a_{n-4} = 0$ $\quad a_0 = 2 \quad a_1 = 0 \quad a_2 = 6 \quad a_3 = 0$

Bellemont 4 Sept 82

Answers Part 1

	Homogeneous Problem	Characteristic Polynomial	Roots
1	$a_n - 2a_{n-1} = 0$	$x-2$	2
2	$a_n - 6a_{n-1} = 0$	$x-6$	6
3	$a_n + 3a_{n-1} = 0$	$x+3$	-3
4	$2a_n - a_{n-1} = 0$	$2x-1$	$1/2$
5	$3a_n + 2a_{n-1} = 0$	$3x+2$	$-2/3$
6	$a_n - a_{n-1} = 0$	$x-1$	1
7	$a_n - 3a_{n-1} = 0$	$x-3$	3
8	$a_n - 2a_{n-1} = 0$	$x-2$	2
9	$a_n + 4a_{n-1} = 0$	$x+4$	-4
10	$a_n + a_{n-1} = 0$	$x+1$	-1
11	$a_n - 5a_{n-1} + 6a_{n-2} = 0$	x^2-5x+6	$2, 3$
12	$a_n - 5a_{n-1} + 4a_{n-2} = 0$	x^2-5x+4	$1, 4$
13	$a_n + a_{n-1} - 6a_{n-2} = 0$	x^2+x-6	$2, -3$
14	$a_n - a_{n-2} = 0$	$x^2 - 1$	$1, -1$
15	$a_n + 5a_{n-1} + 6a_{n-2} = 0$	x^2+5x+6	$-2, -3$
16	$2a_n - 7a_{n-1} + 3a_{n-2} = 0$	$2x^2-7x+3$	$\frac{1}{2}, 3$
17	$6a_n + a_{n-1} - 2a_{n-2} = 0$	$6x^2+x-2$	$1/2, -\frac{2}{3}$
18	$a_n + 4a_{n-1} + 4a_{n-2} = 0$	x^2+4x+4	$-2, -2$
19	$a_n - 6a_{n-1} + 9a_{n-2} = 0$	x^2-6x+9	$3, 3$
20	$a_n + a_{n-2} = 0$	x^2+1	$i, -i$
21	$a_n - 2a_{n-1} - a_{n-2} = 0$	x^2-2x-1	$1 \pm \sqrt{2}$
22	$a_n - 2a_{n-1} + 2a_{n-2} = 0$	x^2-2x+2	$1 \pm i$
23	$a_n - 8a_{n-1} + 15a_{n-2} = 0$	$x^2-8x+15$	$3, 5$
24	$a_n - 3a_{n-1} + 2a_{n-2} = 0$	x^2-3x+2	$1, 2$
25	$a_n - 2a_{n-1} + a_{n-2} = 0$	x^2-2x+1	$1, 1$
26	$a_n - 4a_{n-1} + 4a_{n-2} = 0$	x^2-4x+4	$2, 2$
27	$a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3} = 0$	x^3-3x^2+3x-1	$1, 1, 1$
28	$a_n - a_{n-4} = 0$	x^4-1	$\pm 1, \pm i$
29	$a_n - 5a_{n-1} + 8a_{n-2} - 4a_{n-3} = 0$	x^3-5x^2+8x-4	$1, 2, 2$
30	$a_n - 2a_{n-2} + a_{n-4} = 0$	$x^4 - 2x^2 + 1$	$\pm 1, \pm 1$

Answers Part 2

	General Homo Solution	Forcing function	guess for particular solution
1.	$a_n = A\,2^n$	0	It's homo $a_n = 0$
2.	$a_n = A\,6^n$	3	$a_n = A$
3.	$a_n = A(-3)^n$	$n+2$	$a_n = An + B$
4.	$a_n = A(1/2)^n$	2^n	$a_n = A2^n$
5.	$a_n = A(-2/3)^n$	$3n4^n$	$a_n = An4^n + B4^n$
6.	$a_n = A(1)^n = A$	$6n+10$	$a_n = An^2 + Bn$
7.	$a_n = A\,3^n$	$2 \cdot 3^n$	$a_n = An3^n$
8.	$a_n = A\,2^n$	$-4 \cdot n2^n$	$a_n = An^2 2^n + Bn2^n$
9.	$a_n = A(-4)^n$	$10n^2+1$	$a_n = An^2 + Bn + C$
10.	$a_n = A(-1)^n$	$4n(-1)^n$	$a_n = An^2(-1)^n + Bn(-1)^n$
11.	$a_n = A\,2^n + B3^n$	0	It's homo $a_n = 0$
12.	$a_n = A1^n + B4^n = A + B4^n$	$28n$	$a_n = An^2 + Bn$
13.	$a_n = A\,2^n + B(-3)^n$	0	It's homo $a_n = 0$
14.	$a_n = A + B(-1)^n$	2^n	$a_n = A2^n$
15.	$a_n = A(-2)^n + B(-3)^n$	$72n^2$	$a_n = An^2 + Bn + C$
16.	$a_n = A(1/2)^n + B\,3^n$	$9n2^n$	$a_n = An2^n + B2^n$
17.	$a_n = A(1/2)^n + B(-2/3)^n$	$5n+13$	$a_n = An + B$
18.	$a_n = A(-2)^n + Bn(-2)^n$	45	$a_n = A$
19.	$a_n = A\,3^n + Bn3^n$	8	$a_n = A$
20.	$a_n = A\,i^n + B(-i)^n$	0	It's homo $a_n = 0$
21.	$a_n = A(1+\sqrt{2})^n + B(1-\sqrt{2})^n$	$4(-1)^n$	$a_n = A(-1)^n$
22	$a_n = A(1+i)^n + B(1-i)^n$	$25n3^n$	$a_n = An3^n + B3^n$
23	$a_n = A\,3^n + B\,5^n$	$12 \cdot 3^n$	$a_n = An3^n$
24	$a_n = A + B2^n$	$2n-5$	$a_n = An^2 + Bn$
25.	$a_n = A(1)^n + Bn1^n = A + Bn$	$6n-10$	$a_n = An^3 + Bn^2$
26	$a_n = A2^n + Bn2^n$	$6 \cdot 2^n$	$a_n = An^2 2^n$
27.	$a_n = A1^n + Bn1^n + Cn^2 1^n = A + Bn + Cn^2$	$88n-30$	$a_n = An^4 + Bn^3$
28.	$a_n = A + B(-1)^n + Ci^n + D(-i)^n$	0	It's homo $a_n = 0$
29.	$a_n = A + B2^n + Cn2^n$	2^n	$a_n = An^2 2^n$
30.	$a_n = A + Bn + C(-1)^n + Dn(-1)^n$	0	It's homo $a_n = 0$

Answers Part 3

	Particular solutions	The SOLUTION
1.	$a_n = 0$	$a_n = 3 \cdot 2^n$
2.	$a_n = -3/5$	$a_n = -(6^n) - 3/5$
3.	$a_n = (1/4)n + 5/16$	$a_n = 2(-3)^n + (1/4)n + 5/16$
4.	$a_n = (2/3)2^n$	$a_n = (1/3)(1/2)^n + (2/3)2^n$
5.	$a_n = (6/7)n4^n + (6/49)4^n$	$a_n = (92/49)(-2/3)^n + (6/7)n4^n + (\frac{6}{49})4^n$
6.	$a_n = 3n^2 - 7n$	$a_n = -3 + 3n^2 - 7n$
7.	$a_n = 2n3^n$	$a_n = 10 \cdot 3^n + 2n3^n$
8.	$a_n = -2n^2 2^n + 2n2^n$	$a_n = 5 \cdot 2^n - 2n^2 2^n + 2n2^n$
9.	$a_n = 2n^2 + (16/5)n - 19/25$	$a_n = (31/25)(-4)^n + 2n^2 + \frac{16}{5}n - \frac{19}{25}$
10.	$a_n = 2n^2(-1)^n + 2n(-1)^n$	$a_n = -3(-1)^n + 2n^2(-1)^n + 2n(-1)^n$
11.	$a_n = 0$	$a_n = 3 \cdot 2^n + 2 \cdot 3^n$
12.	$a_n = -2n^2 - (38/7)n$	$a_n = 4 - 4 \cdot 4^n - 2n^2 - (\frac{38}{7})n$
13.	$a_n = 0$	$a_n = 6 \cdot 2^n + 4 \cdot (-3)^n$
14.	$a_n = (4/3)2^n$	$a_n = -1 + (\frac{2}{3})(-1)^n + (4/3)2^n$
15.	$a_n = 6n^2 + 17n + 175/12$	$a_n = 6n^2 + 17n + \frac{175}{12}$
16.	$a_n = -12n2^n - 32 \cdot 2^n$	$a_n = (\frac{1}{2})^n - 3^n - 12n2^n - 32 \cdot 2^n$
17.	$a_n = n + 2$	$a_n = 3(\frac{1}{2})^n - 2(-\frac{2}{3})^n + n + 2$
18.	$a_n = 5$	$a_n = 5(-2)^n - n(-2)^n + 5$
19.	$a_n = 2$	$a_n = 6n3^n + 2$
20.	$a_n = 0$	$a_n = i^n + (-i)^n$
21.	$a_n = 2(-1)^n$	$a_n = (1-\sqrt{2})(1+\sqrt{2})^n + (1+\sqrt{2})(1-\sqrt{2})^n + 2(-1)^n$
22	$a_n = 45n + 18$	$a_n = 2(1+i)^n + 2(1-i)^n + 45n + 18$
23	$a_n = -2n3^n$	$a_n = 3^n - 5^n - 2n3^n$
24	$a_n = -n^2$	$a_n = -2 + 2^n - n^2$
25.	$a_n = n^3 - 2n^2$	$a_n = 2 + n + n^3 - 2n^2$
26.	$a_n = 3n^2 2^n$	$a_n = 5 \cdot 2^n + 3n^2 2^n$
27.	$a_n = n^4 + n^3$	$a_n = 1 + n^3 + n^4$
28.	$a_n = 0$	$a_n = 3 + (-1)^n$
29.	$a_n = n^2 2^n$	$a_n = -n2^n + n^2 2^n$
30.	$a_n = 0$	$a_n = 1 + n + (-1)^n + n(-1)^n$

Chapter 13 Section 1 RELATIONS

Definition of a Relation *1: A relation ~~of~~ on a set X is just a collection of ordered pairs of X.

(An ordered pair is written (a,b) (a & b are elements of the set in question). Note that (a,b) and (b,a) are different ordered pairs. That is $(a,b)=(c,d)$ exactly when $a=c$ and $b=d$.)

Definition of a Relation *2: A relation is a multi-digraph with the ~~properties~~ property that

~~1.~~ there is at most one directed edge from any pair of vertices.

(This means there could be one edge from a to b and an edge from b to a there also could be a loop from a to itself.)

These two definitions are the same*. The set X in 1 coresponds to the vertices of the graph in 2. The ordered pair (a,b) in 1 coresponds to the directed edge from a to b. (Note that (b,a) goes the opposite direction.) (Also (a,a) coresponds to a loop from a to a.)

Notation; Sometimes in Definition 1 the relation is denoted by a capitial letter $(a,b) \in R$, aRb are both used at times to say that a is R-related to b or equivalently in Definition 2 that there is a directed edge from a to b,

Examples all of the following are relations

1. $=$ (equal) on any set
2. $<$ (less than), $>$, $\leq$, $\geq$, $\neq$ on any subset of reals
3. "is a child of", "is a parent of" on trees or people
4. "is redder than", "is bigger than", "is unrelated to" (people)
5. "is a subset of", "is implied by", "is connected to"
6. "are both child of the same parents", "have a parent in common"
7. If $X=\{1,2,3,4\}$ $R=\{(1,1),(1,2),(1,3),(3,1)\}$ is a relation

* Actually sets can be infinite and graph have only a finite number of vertices — but we will allow infinite graphs in this chapter.

4 Properties a relation may have:

A. Reflexive: A relation is reflexive if $\forall a \; aRa$
that is in the graph model <u>all</u> the loops are edges.

B. Symmetric: A relation is symmetric if
$$\forall a \forall b \quad aRb \Rightarrow bRa$$
in the graph model this means if there is an edge from a to b then there is an edge from b to a.
[note that this doesn't require any edge to be there, just that the non-loops come in pairs.]

C. Transitive: A relation is transitive if
$$\forall a \forall b \forall c \text{ if } aRb \text{ and } bRc \text{ then } aRc$$
in the graph model this means if there is an edge from a to b and an edge from b to c then there is an edge from a to c.

D. Anti-symmetric: A relation is anti-symmetric if
$$\forall a \forall b \quad aRb \;\&\; bRa \Rightarrow a = b$$
in the graph model it means if there is an edge from a to b and $a \neq b$ (the edge is not a loop) then there is no edge from b to a.

Examples Relation	Reflexive	Symmetric	Transitive	Anti-symmetric
1 equals	yes	yes	yes	yes
2 less than	no	~~yes~~ no	yes	yes
3 less than or equal to	yes	no	yes	yes
4. is a child of	no	no	no	yes
5. have a parent in common	yes	yes	no	no
6. have both parents in common	yes	yes	yes	no
7. $x - y = 1$ or -1	no	yes	no	no
8. $X = \{1,2,3,4\}$ $R = \{(1,1),(1,2)$ $(1,3)(3,1)\}$	no	no	no	no

3 special kinds of relations:

A. Equivalence Relation: A relation which is reflexive, symmetric and transitive.

(1 & 6 are equivalence relations in the last example)

others include similar or congruent (figures), isomorphic (graphs), give the same remainder when divided by 13.

B. Partial Ordering: A relation which is reflexive, anti-symmetric and transitive

(1, 3 are partial orderings in the last example)

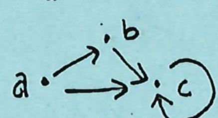
others include "is a subset of" "is an ancestor of" (provided you made everyone his/her own ancestor)

C. Total Ordering: A partial ordering which either has an edge from a to b or b to a.

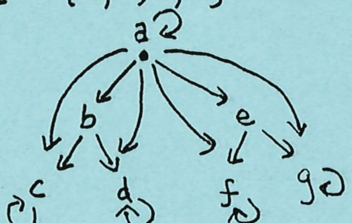
"is a subset of" "is an ancestor of" are <u>NOT</u> total orderings the others in B are.

<u>Exercises:</u> Which of the above ~~examples~~ 7 properties do the following relations enjoy?

1. 

2. 

3. $X = \{1,2,3,4\}$ $R = \{(1,1),(1,2),(2,3),(1,3),(2,2),(3,3)\} (4,4)$

4. 

5. X = real numbers $xRy \iff y \le x+1$

6. "Has a grandparent in common"

7. X = complex numbers $a+bi \le c+di \iff a \le c \;\&\; b \le d$

8. X = complex numbers $a+bi \le c+di \iff$ $(a<c)$ or $(a=c$ and $b \le d)$

9. 

10. " is to the left of " $\underbrace{\text{on books lined up on a shelf}}$

11. " is next to " ditto

12. " is at least as hot as " on stars (like the sun, not people

MORE EXERCISES: 13. A relation is anti-transitive if aRb and bRc imply that ~~a~~ it is not true that aRc.

A. What does this mean in our graph model?

B. Can the graph have any loops? (if it is anti-transitiv

14. A relation is anti-reflexive if aRa is never true.

A. Show that a ordinary graph ~~is~~ canbe thought of as an anti-reflexive symmetric relation. [thinking of the edges as two way paths]

B. Is the reverse also true?

-----

15 If G is any ordinary graph define the following relations on its vertices.

A. " is connected by a path to "

B. " is in the same component as "

C. " is in the same bi-connected component as "

D. " is adjecent to "

E. " has the same color as. (in any $\gamma(G)$-coloring) "

which of the 7 ~~prop~~ propeties these relations have of course depend on the graph G. Which propeties to these relations have for all graphs G? Which are never true for any graph G?

16. Two of the eight possible yes & no combinations for reflexive, symmetric and transitive properties are missing from the examples on Pg 2. Find relations with these combinations of yes and no.

17. A function can also be considered a relation. Indeed define $aFb \Longleftrightarrow f(a)=b$. Since a function is single val! aFb & $aFc \Longrightarrow$ ~~$\$$~~ $b=c$. What does this mean in our graph model?

18. The transitive closure of a relation R is ~~the~~ the relation S where $aSb \Longleftrightarrow$ the are $c_1, c_2, \ldots c_n$ so that either aRb or (aRc_1 and c_1Rc_2 and ... and c_nRb)

A. Show S is transitive.

B. What does this look like for our graph model?

ANSWERS §13.1

Problem	R	T	S	a-S
1	no	no	yes	no
2	no	yes	no	yes
3	yes	yes	no	yes
4	no	yes	no	yes
5	yes	no	no	no
6	yes	no	yes	no
7	yes	yes	no	yes
8	yes	yes	no	yes
9	no	no	yes	no
10	*	yes	no	yes
11	†	no	yes	no
12	$yes	yes	no	$

Equivalence relations: not a single one
partial orders: 3,7,8
total orders: only 8

* is something to the left of itself? I think not, but if true this would be a total order

† is something next to itself? This one is harder to decide but I'll go with no

$ This depends on if there are two suns with exactly the same temperature. This is possible; but if it doesn't happen this is a total ordering.

*15

	R	S	T	aS	E.R.	P.O.	T.O.
A.	sometimes	always	sometimes	sometimes			
B.	always	always	always	sometimes	always		
C.	always	always	sometimes	sometimes		sometimes	
D.	never	always	sometimes	sometimes	never		
E.	always	always	always	sometimes	always		

sometimes ≡ sometimes but not always

most these sometimes are true for a graph with 1 vertex.

Chapter ∞ §1 $O(g(n))$ -notation

If $f(n)$ and $g(n)$ are two functions $: \mathbb{N} \to \mathbb{R}$ so that $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = \infty$. We will say $f(n) = O(g(n))$* if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = L$ and $0 < L < \infty$.

we will say $O(f(n)) \leq O(g(n))$ if $0 \leq L < \infty$.
and $O(f(n)) < O(g(n))$ if $L = 0$.

Note that if $L = \infty$, then $O(g(n)) < O(f(n))$

The Big O notation collects together a bunch of functions that sort of go to infinity at roughly the same speed.

Example 1 $O(n^2) = O(1000n^2) = O(n^2+n) = O(n^2 + \frac{1}{\sqrt{n}})$

for example $\lim_{n\to\infty} \frac{n^2+n}{n^2+\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{n^2(1+\frac{1}{n})}{n^2(1+\frac{1}{n^2\sqrt{n}})} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{1}{n^2\sqrt{n}}} = 1$

Example 2 If $0 < \alpha < \beta$ then $O(n^\alpha) < O(n^\beta)$

for $\lim_{n\to\infty} \frac{n^\alpha}{n^\beta} = \lim_{n\to\infty} \frac{1}{n^{\beta-\alpha}} = 0$ Since $\beta - \alpha > 0$

L'Hospital's Rule (which by the way, was discovered by Bernoulli) If $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = \infty$ and if their derivatives exist $f'(n)$ & $g'(n)$ and if $\lim_{n\to\infty} \frac{f'(n)}{g'(n)} = L$ Then $\lim_{n\to\infty} \frac{f(n)}{g(n)} = L$.

Example 3: If $1 < A < B$ then $O(A^n) < O(B^n)$

for $\lim_{n\to\infty} \frac{A^n}{B^n} = \lim_{n\to\infty} \left(\frac{A}{B}\right)^n = 0$, since $0 < \frac{A}{B} < 1$.

---

* Actually $O(g(n))$ does not require the limit L to exist but this more general situation will not occur here. See Problem 7

Example 4: $O(\ln n) < O(\sqrt{n})$

for $\lim\limits_{n\to\infty} \frac{\ln n}{\sqrt{n}} \underset{\text{by L'H's rule}}{=} \lim\limits_{n\to\infty} \frac{1}{n}\Big/\frac{1}{2\sqrt{n}} = \lim\limits_{n\to\infty} \frac{2\sqrt{n}}{n} = \lim\limits_{n\to\infty} \frac{2}{\sqrt{n}} = 0$

Example 5: $O(n^{100}) < O(2^n)$

for $\lim\limits_{n\to\infty} \frac{n^{100}}{2^n} \underset{L'H}{=} \lim\limits_{n\to\infty} \frac{100\,n^{99}}{2^n \ln 2} \underset{L'H}{=} \lim\limits_{n\to\infty} \frac{100\cdot 99\cdot n^{98}}{2^n(\ln 2)^2} \underset{L'H}{=} \ldots =$

$\lim\limits_{n\to\infty} \frac{100!}{2^n(\ln 2)^{100}} = 0$

Example 6 (more general form of 4) If $\varepsilon > 0$ $O(\ln n) < O(n^\varepsilon)$

for $\lim\limits_{n\to\infty} \frac{\ln n}{n^\varepsilon} \underset{L'H}{=} \lim\limits_{n\to\infty} \frac{1/n}{\varepsilon n^{\varepsilon-1}} = \lim\limits_{n\to\infty} \frac{1}{\varepsilon n^\varepsilon} = 0$

Example 7 (more general form of 6) If $\alpha > 0$ and $\varepsilon > 0$

$O(n^\alpha \ln n) < O(n^{\alpha+\varepsilon})$

for $\lim\limits_{n\to\infty} \frac{n^\alpha \ln n}{n^{\alpha+\varepsilon}} = \lim\limits_{n\to\infty} \frac{\ln n}{n^\varepsilon} = 0$

Example 8 (more general form of 5) If $k > 0$, $A > 1$,

$O(n^k) < O(A^n)$

for $\lim\limits_{n\to\infty} \frac{n^k}{A^n} \underset{L'H}{=} \lim\limits_{n\to\infty} \frac{k\,n^{k-1}}{A^n \ln A} = 0$ by induction or iteration

Example 9 (It does not matter which base of the log we use) If $1 < a \le b$ then $O(\log_a n) = O(\log_b n)$

for $\lim\limits_{n\to\infty} \frac{\log_a n}{\log_b n} \underset{L'H}{=} \lim\limits_{n\to\infty} \frac{1}{n \ln a}\Big/\frac{1}{n \ln b} = \frac{\ln b}{\ln a}$ ← non-zero number.

<u>Theorem 1.</u> Stirling's Formula: If $S(n) = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

then $S(n)\left[1 + \frac{1}{12(n+1)}\right] < n! < S(n)\left[1 + \frac{1}{12(n-2)}\right]$

The proof of Stirling's Formula is more than a bit beyond the reach of most undergraduates. It can be found in OLMSTED, <u>Advanced Calculus</u> p. 490.

Corr 1 $O(n!) = O(S(n))$

pf divide by $S(n)$ & we get $1 + \frac{1}{12(n+1)} < \frac{n!}{S(n)} < 1 + \frac{1}{12(n-2)}$

taking limits we get $1 \le \lim_{n\to\infty} \frac{n!}{S(n)} \le 1$.

Corr 2 $O(n!) = O(\sqrt{n}\,(\frac{n}{e})^n) < O(n^n)$

since $\lim_{n\to\infty} \frac{\sqrt{n}\,(\frac{n}{e})^n}{S(n)} = \frac{1}{\sqrt{2\pi}}$ and $\frac{\sqrt{n}\; n^n/e^n}{n^n} = \frac{\sqrt{n}}{e^n} \to 0$.

Corr 3 $O(\frac{n^n}{e^n}) < O(n!)$

Example 10 if $A > 1$ then $O(A^n) < O(n!)$

for $\lim_{n\to\infty} \frac{A^n}{n^n/e^n} = \lim_{n\to\infty} \frac{(eA)^n}{n^n}$

let this ~~limit~~ limit be z then $\ln z = \ln \lim_{n\to\infty} \frac{(eA)^n}{n^n}$

$= \lim_{n\to\infty} \ln\left(\frac{(eA)^n}{n^n}\right) = \lim_{n\to\infty} n \ln eA - n \ln n = -\infty$

since $\ln z = -\infty$, $z = 0$.

So we get an ordering like

$$O(\ln n) < O(n^\varepsilon) < O(n) < O(n \ln n) < O(n^{1+\varepsilon}) < O(n^2) < \dots$$
$$< O(2^n) < O(5^n) < O(n!) < O(n^n) < \dots$$

Exercises: Show

1. $O(\sqrt{n^2+n+1}) = O(n) = O(n + 100!\sqrt{n})$
2. $O(n^{3/2} + n + \frac{1}{n}) = O(n^{3/2}) = O(n\sqrt{n + 1/n + 100})$
3. $O(\ln(\ln n)) < O((\ln n)^{1/2}) < O(\ln n)$
4. $O(3^n + 2^n) = O(3^n) = O(3^n + n^{100} + 7)$
5. $O(n^3 + \ln n) = O(n^3) = O(100 n^3)$
6. If $O(f(n)) < O(g(n))$ then

   $O(g(n)) = O(4g(n)) = O(g(n) + f(n)) < O(f(n)g(n))$

   (reminber $\lim_{n\to\infty} f(n) = \infty$)
7. Real Definition $f(n) = O(g(n))$ if $\exists\, 0 < A \le B < \infty$ and for large n

   $A g(n) \le f(n) \le B g(n)$. Show $g(n) = n$ $f(n) = g(n) + h(n)$

   satisfy $f(n) = O(g(n))$ when $h(n) =$ [illegible]