GRAPH has no loops nor multiple edges between a pair of vertices. (MKB allows loops!)

A. Definition: A tour T is a sequence of one or more vertices such that $x_i$ is adjacent to $x_{i+1}$ for all i. (path in MKB)

B. Definition: A path P is a tour such that no edge is repeated (in either direction).

C. Definition: A simple path S is a path that doesn't repeat vertices. (endpts can be repeated in MKB)

D. Definition: A circuit C is a path that starts and stops at the same vertex AND uses at least one edge!

E. Definition: A simple circuit is a circuit in which only the starting vertex is repeated. = (cycle in MKB)

1. Disprove: If P, Q are different paths: x to y, then C given by P:x to y followed by Q backwards y to x is a circuit.

2. Lemma: If P, Q are different paths x to y in G, then G has a circuit.

3. Disprove: If T is a tour that starts and stops at x in G then there is a circuit C in G which stops and starts at x.

4. Lemma: If E is an edge from x to y and P is a path x to y which does not use E, then P followed by E is a circuit.

5. Disprove: If P is a path x to y and Q is a path y to z, then P followed by Q is a path x to z.

6. Lemma: If P is a path x to y in G and Q is a path y to z in G then there is a path x to z in G.

6A. Definition: The crossing number of a path is the number of times it comes back to a vertex it has already been to.

6B. Theorem: If there is a path x to y in G, then there is a simple path x to y in G.

6C. Definition: The revisit number of a tour is the number of times it comes back to an edge it has already been to.

6D. Theorem: If there is a tour x to y in G then there is a path x to y in G.

6E. Corollary: If there is a tour x to y in G, then there is a simple path x to y in G.

F. Definition: A tree T is a graph so that for each pair of vertices x,y in T there is a unique path x to y in T.
(note x=y is allowed) [Compare with directed and ordered trees]

7. Theorem: A tree has no circuits.

8. Theorem: Any path P:x to y in a tree is a simple path.

9. Disprove: A graph with no circuits is a tree.

G. Definition: A graph G is connected if for each pair of vertices x,y in G there is a path x to y in G. (weakly connected in MKB)

10. Lemma: A tree is connected.

11. Lemma: A connected graph G with no circuits is a tree.

12. Theorem: A graph T is a tree if and only if it is connected and has no circuits.

H. Definition: If G is a connected graph and the edge E in G is so that if E is removed then G is disconnected, then E is called a cut edge. (also, a bridge in MKB)

13. Disprove: If E is a cut-edge of G and goes x to y, then E is the unique path x to y in G.

14. Lemma: An edge in no circuit is a cut-edge (in a connected graph G).

15. Lemma: A cut-edge is in no circuit.

15'. Disprove: If E is a cut-edge of G and E goes x to y, then E is the only path x to y in G. (but it is the only simple path)

16. Theorem: A graph T is a tree if and only if it is connected and every edge is a cut-edge.

17. Lemma: If E is an edge x to y in a connected graph G, then E is a cut-edge if and only if every path x to y uses E.

18. Lemma: If E is an edge x to y in a connected graph G, then E is a cut-edge if and only if there are vertices w, z in G such that every path w to z uses E.

I. Definition: A directed graph G is said to be a directed tree if there is a vertex called the root and for each vertex x in G there is a unique path which follows the direction of the arrows from the root to x. (x=root is possible)

19. Lemma: In a directed tree, the in-degree of the root is zero.

20. Lemma: In a directed tree, the in-degree of any vertex other than the root is one.

21. Lemma: If in the directed tree T we forget the directions of the edges and that one vertex was special, we have a tree in the usual sense.

22. Lemma: If T is a tree, and we pick one vertex of T to be the root and direct all the edges away from this vertex, we have a directed tree.

23. Lemma: A directed tree has one more vertex than edges.

24. Theorem: A tree with n vertices has n-1 edges.

J. Definition: A subgraph T of a graph G is a spanning tree for G if T is a tree and contains all the vertices of G.

25. Theorem: A graph with a spanning tree is connected.

26. Lemma: If G is a connected, and T is a subgraph of G which is a tree but not a spanning tree of G then there is some edge E of G which is not in T so that if T' is T with E adjoined then T' is a tree.

27. Lemma: If G is a connected graph and H is a connected subgraph of G which is not a tree then there is some edge E of H so that if H' is H with E removed then H' is still connected.

28. Theorem: G is connected if and only if G has a spanning tree.

29. Corollary: If G is connected and has n vertices, then G has at least n-1 edges.

30. Theorem: A connected graph with n vertices and n-1 edges is a tree.

30'. Theorem: A graph with n vertices, n-1 edges and no circuits is a tree (needs K).

31. Theorem: A graph is a tree if and only if it is connected and has fewer edges than vertices.

32. Lemma: A maximal tree subgraph of a connected graph is a spanning tree.

33. Lemma: A minimal connected subgraph containing all the vertices of a connected graph is a spanning tree.

K. Definition: A maximal connected subgraph of G is called a component.

34. Lemma: If G has one component, then G is connected.

35. Lemma: If $C_1$ and $C_2$ are components of G and x is a vertex in both, then $C_1=C_2$.

36. Lemma: If T is a tree, E is any edge in T and S is T with E removed, then S has two components.

37. Lemma: If G has two components $C_1$ and $C_2$ and H is formed by adding an edge from $C_1$ to $C_2$, then H is connected.

38. Lemma: If G is connected, then each cut-edge E is in any spanning tree of G.

39. Theorem: The removal of any cut-edge from a connected graph disconnects the graph into two components.

40. Theorem: A directed tree is planar [strong induction].

41. <u>Corollary:</u> A tree is planar.

42. <u>Theorem:</u> A tree has at least one vertex with degree no more than one.

43. <u>Theorem:</u> A tree with at least one edge, has at least two vertices with degree one.

L. <u>Definition:</u> A vertex x in a connected graph G is an <u>articulation point</u> if H which is formed from G by deleting x and incident edges to x is disconnected

44. <u>Theorem:</u> The vertex x in a tree T is an articulation point if and only if degree(x) is greater than one.

45. <u>Theorem:</u> Each connected graph has a non-articulation point.

45'. <u>Theorem:</u> Each connected graph with at least one edge has two non-articulation points.

46. <u>Proposition:</u> If G is connected and x is a vertex of G, then H (G with x and incident edges removed) consists of a certain number of components say $H_1, \ldots H_n$ (n=1 is possible). Each of the removed edges from G go from x to one of the $H_i$'s and there is at least one edge to each $H_i$.

47. <u>Corollary:</u> If G is connected and x is a vertex of degree n then H has at most n components where H is G with x and incident edges removed.

48. <u>Disprove:</u> A connected graph with a cut-edge has an articulation point.

49. <u>Lemma:</u> A connected graph with at least three vertices and a cut-edge has an articulation point.

50. <u>Disprove:</u> A connected graph with an articulation point has a cut-edge.

51. <u>Definition:</u> A subset S of edges of a connected graph G is called a <u>cut-set,</u> if 1) The graph H formed by deleting the edges in S from G is a disconnected graph and
2) For each subset $S_0$ . S with $S_0 = S$, the graph $H_0$ formed by deleting the edges in $S_0$ from G is a connected graph.

52. <u>Theorem:</u> If G is connected, then for each cut-set S of G and for each spanning tree T of G, S and T have at least one edge in common.

53. <u>Theorem:</u> If G is connected and S is any cut-set of G then there is a spanning tree T of G so that T and S have exactly one edge in common.

54. <u>Theorem:</u> If G is connected and S any cut-set of G, then the graph H formed by deleting the edges of S from G has exactly two components.

55. <u>Theorem:</u> G is connected. For each cut-set S of G and for each circuit C of G, S and C have an even number of edges in common. [Note 0 is even].

56. <u>Theorem:</u> If G is connected and S is a subset of edges such that
1) There is a spanning tree T of G, such that T and S have exactly one edge in common, and

2) S has an even number of edges in common with each circuit C of G
Then S is a cut-set.

57. Lemma: If 1) G is a connected graph
2) T is a spanning tree of G
3) E is an edge in G which isn't in T
4) C is the circuit found in U, the graph obtained by adding E to T
and 5) F is any edge in C and T' is obtained by deleting F from U
then T' is a spanning tree for G.

58. Theorem: If G is a connected graph with weights on each edge, and these weights are distinct then G has a unique minimal spanning tree.

59. Prove: A connected graph G with exactly 1 vertex of degree 1 is not biconnected.

60. Prove: If G is a connected plane graph with R = the number of regions < 40 and E = the number of edges > 65, then there is a vertex of G of degree less than or equal to 4.

61. Prove: If the graph G has 11 vertices, prove that either G or G is nonplanar.

62. Prove: If every pair of vertices of graph G lie on a common simple circuit, then G is biconnected. (False if simple is removed)

63. Prove: If the 63 chairs of a classroom are arranged into 7 rows of 9 chairs each (and 9 columns of 7 each) show that if 30 students are seated in the classroom, thene some row has greater than or equal to 5 students and some column has less than or equal to 4 students.

64. Prove: Prove that every connected graph containing only vertices of even degree has no cut-edge.

65. Prove: Suppose that 88 chairs are arranged in a rectangular array of 8 rows with 1 chairs each. If 50 students are seated (1 student per chair) then prove that some row must have at least 7 students and some column must have at most 4 students.

66. Prove: Suppose that G is a connected plane graph such that the number of regions is less than 52 and the number of edges is greater than 85. Prove that there is a vertex v of G such that deg(v) is less than or equal to 4.