COT 3132 Boolean Algebras

A <u>Boolean Algebra</u> is a collect of things B together with two binary operations $+$ and $\cdot$ such that (0) – (5) & (0′) – (5′) are true.

(0) $a, b \in B \Rightarrow a+b \in B$ (0′) $a, b \in B \Rightarrow a \cdot b \in B$

These are the closure properties.

(1) $a, b \in B \Rightarrow a+b = b+a$
(1′) $a, b \in B \Rightarrow a \cdot b = b \cdot a$
commutativity

(2) $a, b, c \in B \Rightarrow (a+b)+c = a+(b+c)$
(2′) $a, b, c \in B \Rightarrow (a \cdot b) \cdot c = a \cdot (b \cdot c)$
associative

(3) $a, b, c \in B \Rightarrow a \cdot (b+c) = a \cdot b + a \cdot c$
(3′) $a, b, c \in B \Rightarrow a + b \cdot c = (a+b) \cdot (a+c)$
distributi

There are <u>two</u> special elements 0 and 1 in B $(0 \neq 1)$ s.t.

(4) $a \in B \Rightarrow a+0 = a$ (4′) $a \in B \Rightarrow 1 \cdot a = a$

For each $a \in B$ there is an $a' \in B$ (a complement) so that (5) $a + a' = 1$ (5′) $a \cdot a' = 0$

~ END OF DEFINITION OF A BOOLEAN ALGEBRA.

There are <u>many</u> difference Boolean Algebra's, but the example to keep in mind is the following one. Let X be any non-empty set and let $B = \mathcal{P}(X)$ the set of all subsets of X. If $a, b \in B$, $a+b$ is $a \cup b$, $a \cdot b = a \cap b$, a' is a^c (the set of things not in a), 0 is $\emptyset$ and 1 is X.

Boolean algebras satisfy many other identities but these follow from those above (i.e. can be proved). We list them as Theorems below. (Actually (2) & (2′) follow from the rest also)

Uniqueness of 0, 1:

(6) The is only one element $0 \in B$ s.t. $a \in B \Rightarrow a + 0 =$
(6′) The is only one element $1 \in B$ s.t. $a \in B \Rightarrow 1 \cdot a =$

Uniqueness of complements:

(7) For $a \in B$ there is only one $a' \in B$ s.t. both
$$a + a' = 1 \text{ and } a \cdot a' = 0$$

Idempotence:

(8) $x \in B \Rightarrow x + x = x$ (8') $x \in B \Rightarrow x \cdot x = x$

[Note we cannot cancel $0+1=1$ by (4) and $1+1=1$ by (8) hence $0+1=1+1$ but $0 \neq 1$ by (4)(4')]

[Similarly $0 \cdot 1 = 0 \cdot 0$ but $1 \neq 0$]

More properties of 0 & 1: if $x \in B$ then (Domination)

(9) $1 + x = 1$ (9') $0 \cdot x = 0$

Double complementation: (Involution)

(10) $a \in B$ then $(a')' = a$

Absorption: $a, b \in B$ then

(11) $a + a \cdot b = a$ (11') $a \cdot (a+b) = a$

Since More on 0 & 1:

(12) $0' = 1$ (12') $1' = 0$

This one doesn't seem to have a name, but it is most useful.

If $x, y \in B$ then (13) $x \cdot (x'+y) = xy$ (13') $x + x'y = x + y$

DeMorgan's Laws: if $x, y \in B$ then

(14) $(x+y)' = x'y'$ (14') $(xy)' = x' + y'$

Consensus: $x, y, z \in B$ then

(15) $xy + x'z + yz = xy + x'z$

(15') $(x+y)(x'+z)(y+z) = (x+y)(x'+z)$

~ End of 1st collection of Theorems. The above should all become "hard wired" into your brain. However there are many more formula's of interest here are some:

The operation $\oplus$: if $a, b \in B$ define $a \oplus b = ab' + a'b$

(16) $a, b \in B$ then $a \oplus b = b \oplus a$

(17) $a, b, c \in B$ then $(a \oplus b) \oplus c = a \oplus (b \oplus c)$

(18) $a,b,c \in B \Rightarrow a(b \oplus c) = (ab) \oplus (ac)$

(19) $a,b \in B \Rightarrow a \oplus b = (a') \oplus (b')$

(20) $a \oplus a = 0$; $0 \oplus a = a$; $1 \oplus a = a'$

(21) $a \oplus (a \oplus b) = b$

(22) If $a \oplus b \oplus c = d$ then $a \oplus b = c \oplus d$ and $a = b \oplus c \oplus d$

Cancellation rules (not as useful as they look)

(23) $x + a = x + b$ & $xa = xb \Rightarrow a = b$

(24) $x + a = x + b$ & $x' + a = x' + b \Rightarrow a = b$

(24') $xa = xb$ & $x'a = x'b \Rightarrow a = b$

An ordering   Define $a \leq b \Leftrightarrow a \cdot b = a$

(25) $a \in B \Rightarrow a \leq a$ (reflexivity)

(26) $a,b \in B$ $a \leq b,\ b \leq a \Rightarrow a = b$ (anti-symmetry)   partia orderi

(27) $a,b,c \in B$ $a \leq b,\ b \leq c \Rightarrow a \leq c$ (transitivity)

(28) $x \in B \Rightarrow 0 \leq x \leq 1$

(29) $a,b \in B \Rightarrow ab \leq a \leq a+b$ lattice propert

(30) $a,b,c \in B$ $a \leq c$ & $b \leq c \Rightarrow a+b \leq c$

(30') $a,b,c \in B$ $c \leq a$ & $c \leq b \Rightarrow c \leq a \cdot b$

(31) $a,b,c \in B$ $a \leq b \Rightarrow a+c \leq b+c$

(31') $a,b,c \in B$ $a \leq b \Rightarrow ac \leq bc$

(32) $a,b \in B$ $a \leq b \Longleftrightarrow a+b = b$

(33) $a \leq a' \Longleftrightarrow a = 0$ (33') $a' \leq a \Leftrightarrow a = 1$

If $\leq$ satisfies (25), (26) & (27) it is called a partial ordering. If a,b then $\sup(a,b)$ is the element (if it exists) s.t. $a \leq \sup(a,b)$, $b \leq \sup(a,b)$ and if $a \leq c$ & $b \leq c$ then $\sup(a,b) \leq c$. $\inf(a,b)$ is s.t. $\inf(a,b) \leq a$, $\inf(a,b) \leq b$ and if $d \leq a$ & $d \leq b$ then $d \leq \inf(a,b)$.

(34) $\sup(a+b) = a+b$ (34') $\inf(a,b) = a \cdot b$