COMPLEX ANALYSIS 19 May 1970

Problem Set

I. Let Ω be the union of all open polydiscs $\Delta(0;r)$ on which $f(z) = \sum_{\alpha} c_{\alpha} z^{\alpha}$ converges.

(A) Prove that z belongs to Ω if and only if $f(z)$ converges absolutely on a neighborhood of z.

(B) Prove that $f(z)$ converges uniformly on compact subsets of Ω.

(C) What can you say about the geometry of Ω? Is it star-shaped? Convex? Circled?

(D) What can you say about the convergence of the series $D^{\alpha}f$.

II. Proof the best identity theorem you can.

III. Suppose f is analytic on $\Delta(0;1)$ and continuous of $\overline{\Delta}(0;1)$ and suppose $|z_n| = 1$ is fixed. Let $g(z_1, z_2, \ldots, z_{n-1}) = f(z)$. Prove that g is analytic on $\Delta(0;1)$ in $C^{(n-1)}$.

IV. (A) Prove that a pluriharmonic function on a polydisc is the real part of an analytic function.

(B) Find an analytic function whose real part is $e^x(x \cos y - y \sin y) + x^3 - 3xy^2$.

TEXT: 10-7

V. Use the Weirstrass preparation theorem to prove the implicit function theorem for the simple case $f(z_1, z_2, \ldots, z_n) = 0$ near a point where $f(z) = 0$ but $\partial f/\partial z_n \neq 0$.

COMPLEX ANALYSIS
Math 333

Problem Set I
Due Oct. 20, 1972

I. Consider the function $T(z) = (z-a)/(z-\bar{a})$, where a is not real. If z is real, prove that $|T(z)| = 1$. If the imaginary parts of z and a are both positive, what can you say about $|T(z)|$? Prove it.

II. (A) Suppose $a = re^{i\theta}$ is a non-zero complex number. Find a formal power series $f(X)$ for which $g(z) = f(z-a)$ is a branch of $\log z$ in the disc $|z-a| < r$. [Hint: Recall that you need only show that $g'(z) \equiv 1/z$ and that $\exp(g(a)) = a$.]

(B) Prove that there is an analytic branch of $\log z$ in the right half-plane, $\mathrm{Re}(z) > 0$. Prove that there is a branch of $\log z$ in the complement of any closed half-line through the origin, but that there is no branch of $\log z$ in the complement of the origin.

III. (A) Let $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences of numbers with the following properties:

(i) there is a constant $M > 0$ such that

$$|\alpha_1 + \alpha_2 + \cdots + \alpha_n| \leq M \text{ for all } n \geq 1,$$

(ii) the β_n are real ≥ 0 and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n \geq \cdots$.

Show that, for all $n \geq 1$,

$$|\alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_n\beta_n| \leq M\beta_1.$$

(Hint: Introduce $s_n = \alpha_1 + \cdots + \alpha_n$ and write

$$\alpha_1\beta_1 + \cdots + \alpha_n\beta_n = (\beta_1-\beta_2)s_1 + \cdots + (\beta_{n-1}-\beta_n)s_{n-1} + \beta_n s_n.)$$

(B) For which values of z does the series $\sum z^n/n$ converge? Pay particular attention to $|z| = 1$.

IV. If $\{a_n\}$ is a sequence of positive numbers, prove that

$$\underline{\lim}\, \frac{a_{n+1}}{a_n} \leq \underline{\lim}\, (a_n)^{1/n} \leq \overline{\lim}\, (a_n)^{1/n} \leq \overline{\lim}\, \frac{a_{n+1}}{a_n}.$$

V. (A) Prove that $f(z) = \cos z$ is the unique analytic function for which $f''(z) = -f(z)$ and $f(0) = 1$, $f(1) = 0$. State and prove similar theorems for $\cos z$ and e^z.

(B) Use (A) to prove $\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$ and $e^{\lambda+z} = e^{\lambda}e^{z}$.

VI. If $f = \sum_0^\infty a_n X^n$, let $N(f) = \sum_0^\infty \frac{|a_n|}{1+|a_n|}$.

(A) Prove that $C[[X]]$ becomes a complete metric space if we define the distance between f and g as $N(f-g)$. This topology on $C[[X]]$ is called the *Fréchet* topology.

(B) Prove that a sequence, $\{f_k\}$, of formal power series converges to g in the Fréchet topology if and only if the n'th coefficient of f_k converges to the n'th coefficient of g, for all n. [Remark: this is just another way of saying that the Fréchet topology on $C[[X]]$ is the same as the product topology when $C[[X]]$ is considered as a product of countably many copies of the complex numbers.]

(C) Prove that the polynomials are dense in the Fréchet topology on $C[[X]]$.

(D) Suppose that $\{S_k(X)\}$ is a summable family of power series with $S(X) = \sum_{k=1}^\infty S_k(X)$. [See Cartan, p. 11, for the definition of summable and of $S(X)$.] Prove that $\lim_{m\to\infty} \sum_{k=1}^m S_k(X)$ is $S(X)$ in the Fréchet topology. Find a series of elements in $C[[X]]$ which is *not summable* in Cartan's sense, but which converges in the Fréchet topology. [Remark: for a topology in which convergence and summability are identical, see problem 1, p. 43, in Cartan.]

VII. (A) Prove that $C[[X]]$ is a topological algebra in the Frechet topology (i.e., prove that addition, multiplication, and scalar multiplication are all continuous). [Hint: *Never* use the metric to prove continuity; use VI(B) instead.]

(B) Let g be a power series with zero constant term and define $\Phi(f) = f \circ g$. Prove that Φ is continuous. Use the continuity to prove that Φ is an algebra homomorphism of $C[[X]]$.

(C) Suppose that ψ is a continuous algebra homomorphism of $C[[X]]$ and that $\psi(z) = g$. Prove that g has zero constant term. [Hint: $\lim X^n = 0$], and that $\psi(f) = f \circ g$, for all f. Use these results to prove the associativity of formal power series composition (i.e., to prove proposition 4.1, p. 13, in Cartan).

<u>DO THE FOLLOWING PROBLEMS FROM AHLFORS *Complex Analysis*:</u>

<u>Page</u>	9	Problem	3
"	11	"	1
"	16	"	2,3,4,5
"	41	"	1,2,3,5,6
"	45	"	1,2,3
"	48	"	3,4,6
"	182	"	3

REFERENCES ON SEVERAL COMPLEX VARIABLES

SURVEY

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C. Caratheodory, *Theory of Functions*, 2nd English Edition, Chelsea; Volume II, pp. 113-128.

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COMPLEX ANALYSIS
Problem Set

2

Due March 17, 1970

1

TEXT: ~~11.1, 11.x, 12.x, 12.x, 13.x, 13.x, 13.x.~~

started

I. Discuss symmetry with respect to circles and lines and reflections of meromorphic functions. You may assume Theorem 11.17, but fill in the details of other proofs sketched in class.

II

I claim if $d\mu = h\,d\theta$, $h \in L^\infty$ then I could let $g \in L^1$

Prove if $h \in L^\infty$ and if $f_r(\theta) = P(h)$ ~~$re^{i\theta}$~~ then $f_r \to h$ weak* on $L^\infty = (L^1)^*$

$$\lim_{r \to 1} \int f_r(\theta)\, g(\theta)\, d\theta \to \int h(\theta)\, g(\theta)\, d\theta \quad \forall g \in L^1$$

III

L^∞

Case

$|z| = 1 \Rightarrow |f(z)| = 1$

$\boxed{5, 6}$

$$\sum_{n=0}^{\infty} a_n z^n$$

$e^{i\theta} \to e^{i\phi}$

$\theta \to \phi$

$S^1 \to S^1$

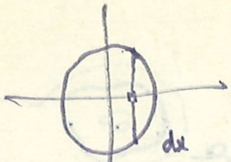
$$\sum_{n=0}^{\infty} a_n = e^{i\theta}$$

$$\sum_{n=0}^{\infty} (-1)^n a_n = 1$$

$f(z) \to e^{i\theta}$ for some

$f(z) \to e^{i\theta} z$ for some θ

$f(z) \to e^{i\theta} z^n$ for some θ some $n \geq 0$



$u_x = \frac{1}{\pi r^2} \iint_{D(a,r)} u_x \, dx\,dy$

$u = \int_0^x u_x \, dx \qquad \iint$

COMPLEX ANALYSIS
Problem Set

Due March 3, 1970

$u(x,y) = \int_0^x u_x(x,y)\,dy \qquad u_x(0) = \frac{1}{\pi r^2} \iint_{D(0,r)} \cancel{u_x(x,d}$

TEXT: 11-3, 11-6, 11-7 (Hint: It will be enough to consider discs), 11-18 (Hint: Consider discs first and use 11-6 (b)).

$u_x(r$

I. A function is said to be locally constant if every point contains a neighborhood on which the function is constant. Prove that a locally constant function with connected domain is constant. If f and g are analytic functions with identical real parts, prove that f-g is locally constant.

II. Prove that if u is real harmonic in the entire plane and is either bounded below or above, then u is constant.

III. Show that a continuous function satisfies the formula of Problem 11-5 if, and only if, it is harmonic.

IV. A) Derive the polar form of Laplace's Equation.

B) Suppose that u is harmonic in an annulus about the origin and that

$$f(r) = \frac{1}{2\pi} \int_0^{2\pi} u(r,\theta)\, d\theta.$$

$R > \frac{1}{2}$

Prove that $\frac{d}{dr}(r f'(r)) = 0$. What is f(r)?

$\mathrm{Re}[e^z] = e^{\alpha+\beta i}$

C) Use Part B) to prove the mean value property for harmonic functions.

D) Use Part B) to show that a function which is bounded and harmonic in a deleted neighborhood of a point can be extended to a harmonic function in the entire neighborhood.

$e^{\alpha\cos\beta}$

$e^{\frac{1}{2}}$

V. Suppose that u is harmonic in the complement of D(0,1-ε) and that u(∞) = 0. Find a Poisson integral type formula for u.

$\sqrt{e}$

$\frac{R-r}{R+r}\, u(a) \le u(a + re^{i\theta}) \le \frac{R+r}{R-r}\, u(a)$

$\frac{1}{1-\frac{1}{2}}$

$\frac{1}{2} < R$

$\frac{1}{3} \qquad \frac{\epsilon}{1+\epsilon}\ \cancel{1}\ 1 \le u(0 + \tfrac{1}{2}e^{i\theta}) \le \frac{1+\epsilon}{\epsilon} \qquad \frac{3/2}{1/2} = 3$

COMPLEX ANALYSIS
Math 333

Problem Set #3
Due Nov. 17, 1972

XIV. (A) Suppose that $g(z,t) = g(x+iy,t)$ is a function from an open subset of $C^1 \times R = R^3$ to C^1. Suppose that $\partial g/\partial z$ and $\partial g/\partial t$ are continuous. Assuming the theorem about real differentiation under integral signs, prove that

$$F(z) = \int_a^b g(z,t)dt \text{ has complex derivative } F'(z) = \int_a^b \frac{\partial g}{\partial z}(z,t)dt.$$

(B) If Ω is an open subset of C^1 which is star-shaped about z_0 and $f(z)$ has a continuous derivative throughout Ω, prove that $f(z)dz$ is exact in Ω. [Hint: For each z in Ω, let γ_z be the straight line from z_0 to z; and define:

$$F(z) = \int_{\gamma_z} f(z)dz = \int_0^1 f(z_0+t(z-z_0))(z-z_0)dt.$$

Show by (A) that $F'(z) \equiv f(z)$ in Ω.]

XV. Suppose $f(z)$ has a continuous derivative on the annulus $r < |z-a| < R$. Let C_1 and C_2 be positively oriented circles centered at a and with radii between r and R. Prove that

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

[Hint: Partition the annulus into a finite collection of star-shaped sectors.]

XVI. A differential form $\omega = P(x,y)dx + Q(x,y)dy = g(z)dz + h(z)d\bar{z}$ is said to be closed in an open set Ω if P and Q have continuous partial derivatives and $\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}$.

(A) Find conditions on g and h which are equivalent to ω being closed.

(B) Show that $f(z)dz$ is closed if and only if $f'(z)$ exists and is continuous.

(C) If ω is closed in a star-shaped region Ω, prove that it is exact. [Notice that this generalizes XIV(B).]

(D) Find a statement of some form of Green's Theorem in the plane. Rewrite this theorem, translating the formulas involving P,Q and their x,y derivatives to formulas involving g,h and their $z,\bar{z}$ "derivatives".

DO THE FOLLOWING PROBLEMS IN AHLFORS:

P. 108	Problems 2,3,4,5,6,7
P. 117f	Problems 2,3
P. 120	Problems 1,2,3

Done

$$\frac{1}{2\pi}\int_0^{2\pi}\left[\left(\frac{1+z}{1-z}\right)^{\frac{1}{p}}\right]^p d\theta \qquad \int_{C_r}\left|\frac{1+re^{i\theta}}{1-re^{i\theta}}\right| = 1$$

$|z|=r$

COMPLEX ANALYSIS
Problem Set 3

Due April 13, 1970

For H^p spaces always assume $1 \le p \le \infty$; and for $p = 1$ you may use Theorem 17.13.

$$f = P(f^*) . \quad f^* = \mathrm{Re}^+ f - \mathrm{Re}^- f + \mathrm{Im}^+ f - \mathrm{Im}^- f$$

I. If $f \in H(U)$, prove that $f \in H^1(U)$ if and only if f is a linear combination of positive harmonic functions.

Started

~~II. Find a measure μ for which $F_\mu(\theta) = F(2\theta)$. Do Problem 11-8.~~
(Hint: First do 11-10 as a lemma; you may assume Theorem 8.9.)

TEXT: ~~13-11(d)~~, ~~14-9~~, [redacted], ~~17-1~~, ~~17-3~~, ~~17-12~~. ~~13-4~~, ~~13-5~~

Started

Started

III Let K be Compact V one component of S^2-K and suppose $\alpha=\infty\in V$ ($\alpha\neq\infty$ $\frac{1}{z-\alpha}$) cannot be uniformly approximate on K by rational function with poles off $K\cup V$.

$$\frac{1+2z+z^2}{1-2z+z^2}$$

$$\frac{1-r^2+r(e^{i\theta}-e^{-i\theta})}{1-2r\cos\theta+r^2} \quad \left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right)\left(\frac{1-re^{-i\theta}}{1-re^{-i\theta}}\right)$$

$$\frac{1+2\bar z - \bar z^2}{1-2\bar z+\bar z^2}$$

$$\left[\frac{1-r^2+ir\sin\theta}{1-2r\cos\theta+r^2}\right]^{\frac{1}{p}}$$

$$\frac{re^{i\theta}}{re^{-i\theta}} \qquad re^{i4\theta}$$

$z \in \overline{D-\epsilon}$, $f_n(z) - f_m(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f_n(\xi)-f_m(\xi)}{\xi - z}\, d\xi$ $\le M$

$\frac{1}{|\xi - z|} \le K$

$\frac{\partial G}{\partial x} = \frac{\partial G}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial G}{\partial \theta}\frac{\partial \theta}{\partial x}$

COMPLEX ANALYSIS Due April 28, 1970

Problem Set 4

$|f_n(z) - f_m(z)| \le \frac{K}{2\pi}\int |f_n(\xi) - f_m(\xi)|\, d\xi$

TEXT: ~~10-17~~, ~~13-7~~, 13-10, ~~13-14~~, ~~14-10~~, 20-1, 20-2.

started started started

I. Prove Green's theorem for the annulus bounded by $|z| = a$ and $|z| = b$. HINT: Let C_r be the circle of radius r and A_r the annulus bounded by $|z| = a$ and $|z| = r$. Then define:

$$f(r) = \int_{C_r} P\,dx + Q\,dy - \int_{C_a} P\,dx + Q\,dy$$

$$g(r) = \iint_{A_r} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Prove that $f'(r) = g'(r)$ and $\boxed{f(a) = g(a) = 0.}$

II. Let $F(z) = z + 1/z$. Prove that $1/F$ belongs to $\mathcal{S}$ and find the range of F. For which numbers a does $1/(F-a)$ belong to $\mathcal{S}$? Use F and $1/(F-a)$ to show that Bieberbach's conjecture and the estimates of sections 14.10 through 14.15 are optimal. What is the analytic capacity of a line segment of length r?

started