2. **Thm**: If $f'(x)$ exists at $x=a$, then $f(x)$ is continuous at $x=a$.

Proof (?)

I. We prove the theorem by contradiction.

II. Suppose $f(x)$ is not continuous at $x=a$ then there is an $\epsilon>0$ such that for $\delta>0$ $|x-a|<\delta$ implies $|f(x)-f(a)| \ge \epsilon$.

III. Now for $|x-a|<\frac{1}{N}$ we have $\frac{1}{|x-a|} > N$ and $\left|\frac{f(x)-f(a)}{x-a}\right| \ge N\epsilon$

IV. But this implies $\lim_{x\to a}\left|\frac{f(x)-f(a)}{x-a}\right| = +\infty$

V. $|f'(a)| = \left|\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right| = \lim_{x\to a}\left|\frac{f(x)-f(a)}{x-a}\right| = +\infty$ because $|x|$ is a continuous function and IV

VI. But this is a contradiction for $f'(x)$ exists $x=a$ means $f'(a)$ is not infinite.

3. **Thm** A monotone increasing sequence $\{a_n\}$ either converges or has limit $=+\infty$. In either case $\lim_{n\to\infty} a_n = \sup\{a_n\}$.

Proof (?)

I. By the fact $\sup\{a_n\}$ is a upper bound $\forall n \; a_n \le \sup\{a_n\}$.

II. Since $\sup\{a_n\}$ is a sup, for all $\epsilon>0$ there is an N with $a_N > \sup\{a_n\}-\epsilon$.

III. Since $\{a_n\}$ is monotone for $n>N$ $a_n \ge a_N > \sup\{a_n\}-\epsilon$

IV. Thus for $n>N$ $\sup\{a_n\} \ge a_n > \sup\{a_n\}-\epsilon$

V. Therefore $\lim_{n\to\infty} a_n = \sup\{a_n\}$

VI. Since $\sup\{a_n\}$ always exists $\{$if we allow $\pm\infty\}$ the second sentence in the theorem, implies the first $\{$since $\sup\{a_n\}$ can't be $=-\infty\}$ sentence in the statement of the theorem.

4 <u>Thm</u>: If S and T are non-void sets of positive reals and if $S+T = \{s+t;\ s\in S, t\in T\}$, then $\sup S + \sup T = \sup(S+T)$.

<u>proof (?)</u>

I. Since $\sup S$ (respectively $\sup T$) is an upper bound for S (respectively T) we have $\forall s\in S\ s\le \sup S$ (respectively $\forall t\in T\ t\le \sup T$).

II. Hence $s+t \le \sup S + \sup T$ for $s\in S, t\in T$.

III. Since $\sup S + \sup T$ is an upper bound for $S+T$ we have $\sup(S+T) \le \sup S + \sup T$.

IV. Now we break the prove of the reverse inequality into two cases: (A) one of $\sup S$ or $\sup T$ is $+\infty$; (B) neither $\sup S$ nor $\sup T$ is $+\infty$.

V. To complete the proof for case A we show $\sup(S+T) = +\infty$

VI. Suppose $\sup S = +\infty$, thus for each $n\in \mathbb{N}$ there is an $s\in S$ with $s\ge n$

VII. Thus for any $n\in\mathbb{N}$ and $t\in T$ there is an $s\in S$ with $s+t \ge n$.

VIII. Therefore $\sup(S+T) = +\infty$.

IX. Case B: Let $s_0\in S$ s.t. $s_0 = \sup S$ and $t_0\in T$ with $t_0 = \sup T$.

X. Since $s_0+t_0 \in S+T$, $s_0+t_0 \le \sup(S+T)$.

IX. Thus $\sup S + \sup T = s_0+t_0 \le \sup(S+T)$

IIX. Since we have shown one inequality in III and the other in <u>VIII</u> & <u>IX</u>, we have $\sup S + \sup T = \sup(S+T)$.

5 Thm: A monotone function is continuous except at a finite number of pts on any bounded interval.

proof(?)

I. We may assume $f(x)$ is an increasing function on the interval $[0, 1]$

II. Hence $f(0)$ is the minimum value and $f(1)$ is the maximul value of $f(x)$ on $[0, 1]$,

III. Thus for each $a \in (0,1)$, $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ exists because the function

is increasing (or decreasing for $x \to a^+$) and bounded above (respectively bounded below)

IV. Suppose $f(x)$ is not continuous at $x = a$, but the lim $f(x)$ does exist (i.e. both limits in III are equal)

Then $\lim\limits_{x \to a} f(x) \neq f(a)$ [say $\lim\limits_{x \to a} f(x) < f(a)$] then lim $f(x) < f(a)$ (the other case is similar)

V. But this is impossible since it implies the are $x > a$ with $f(x) < f(a)$ which contradicts the fact f is increasing,

VI. So if $f(x)$ is not continuous at $x = a$ then $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$, and by the increasing natural of $f(x)$ we may concluded $\lim\limits_{x \to a^-} f(x) < \lim\limits_{x \to a^+} f(x)$,

Some definitions let $f(a^-) = \lim\limits_{x \to a^-} f(x)$, $f(a^+) = \lim\limits_{x \to a^+}$ and let $J(a) = f(a^+) - f(a^-) \sim$ the jump at $x = a$.

VII. To complete the proof it suffices to show that there are only finitely many jumps.

VIII. Let $\epsilon > 0$ and let $0 < a_1 < a_2 \dots a_n < 1$ be points where $J(a_i) \geq \epsilon$. Then

$f(1) - f(0) = (f(1) - f(x_{n+1})) + (f(x_{n+1}) - f(x_n)) + \dots + (f(x_2) - f(x_1))$
$+ (f(x_1) - f(0))$ where $0 < x_1 < a_1 < x_2 < a_2 \dots x_n < a_n < 1$

IX. Now $f(x_{i+1}) - f(x_i) \geq f(a_i^+) - f(a_i^-) = J(a_i) \geq \epsilon$.
Hence $f(1) - f(0) \geq n\epsilon$ or $\dfrac{f(1) - f(0)}{\epsilon} \geq n$.

X. Therefore there are only finitely many discontinuities since ϵ was arbitrary.

Thm Consider (1) $\lim_{n\to\infty} a_n$, (2) $\lim_{n\to\infty} b_n$ and (3) $\lim_{n\to\infty}(a_n+b_n)$. If any two of (1), (2) or (3) exist, then the other also exists and in this case (1) + (2) = (3) [exists means converges or has a limit in $\mathbb{R}$]

Proof (?)

I. If (1) & (2) exist implies that (3) exists is a theorem in the book.

II. Otherwise, by symmetry, we complete the proof by showing (1) & (3) existing implies the existence of (2). [Actually we need statement X as well.]

III. Suppose (2) does not exist, then (b_n) is not a Cauchy sequence.

IV. So there is an $\epsilon > 0$ and a subsequence (b_{n_k}) of (b_n) such that $|b_{n_k} - b_{n_{k+1}}| \geq \epsilon$, for $k = 1, 2, 3, \ldots$

V. One of three things can happen

A: (b_{n_k}) has a subsequence whose limit is $+\infty$

B: (b_{n_k}) has a subsequence whose limit is $-\infty$

C: (b_{n_k}) has two subsequences $(b_{n_{k_j}})$ and $(b_{n_{k_i}})$ with $\lim_{j\to\infty} b_{n_{k_j}} = B$ and $\lim_{i\to\infty} b_{n_{k_i}} = B^*$ and $B \neq B^*$

VI. In case A & B imply $(a_n + b_n)$ has a subsequence whose limit is one of the infinities, a contradiction.

VII. In case C, let $\lim_{n\to\infty} a_n = A$ then $\lim_{i\to\infty} a_{n_{k_i}} + b_{n_{k_i}} = A + B^*$ and $\lim_{j\to\infty}(a_{n_{k_j}} + b_{n_{k_j}}) = A + B$.

VIII. Hence $\lim_{n\to\infty}(a_n + b_n)$ does not converge a contradiction

IX. There existence of (1) & (3) $\Rightarrow$ (2) exists

X. Finally (1) + (2) = (3) follows from the theorem in book given in I.