EXERCISES - Alg topology - 10/7/69 - Williamson

Due Oct 20

Week from Mon

1. Show that an interval does not have the homotopy extension (HEP) property relative to an open subinterval - say the pair $([0,1], [0,\frac{1}{2}))$. (for ~~not~~ every space Y)

2. Suppose (X,A) has the HEP. Let X/A be the quotient space formed by identifying A to a base point x_0 ($x_0 \in A$). Then show that $X \simeq X/A$. (add to hyp: A is contratibile (rel x_0))

3. Show that, if (A,B) has the HEP and (X,A) does, then (X,B) does.

$f : X \times \{0\} \cup A \times I \to Y$ $f =$ iden $Y = X \times \{0\} \cup A \times I$

$\exists \tilde{f} : X \times I \to Y$

4. Show that if (X,A) has the HEP (for every space Y, as above) then there is a retraction $X \times I \to X \times 0 \cup A \times I$.

5. Assume that (D^n, S^{n-1}) has the HEP ($D^n = \{x \in R^n \mid |x| \leq 1\}$), which is true. Let $f: S^{n-1} \to A$ be given, and let $X = A \cup_f D^n$ be the quotient space of the disjoint union $A \cup D^n$ by identifying $x \in S^{n-1}$ with $f(x) \in A$. Show that (X,A) has the HEP.
show $\exists$ retraction

6. Suppose $f: (X, x_0) \to (Y, y_0)$ is a base point preserving map and X is normal (Kelly, General Topology, 113-115) show that $f \simeq$ constant map at y_0 in the base point sense iff it is without the restriction of holding x_0 fixed.
$\Rightarrow$ ✓

x_0 closed contractable resp x_0 neighbor normal $f \simeq y_0$

F $X \times I$ $f = F(x,1)$ $y_0 = F(x,0)$

Notes- Algebraic Topology.

We say that (continuous) maps $f, g: X \to Y$ between topological spaces are <u>homotopic</u> iff there is a map $F: X \times I \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$. We also say f can be <u>deformed</u> into g, and call F a <u>homotopy</u>, or a <u>deformation</u>.

If $A \subset X$, $B \subset Y$
We may also consider a map of pairs $f: (X,A) \to (Y,B)$, i.e. a map $f: X \to Y$ such that $f(A) \subset B$, and we may require of a homotopy that $F(x,t) \in B$ for all $x \in A$, $t \in I$; we then say $f \simeq g$ (relative to A), or $f \simeq g$ as maps of pairs. If A consists of a single point x_0 and B of a point y_0, which we call <u>base points</u>, we speak of base point preserving homotopies.

THM Homotopy is an equivalence relation, and the equivalence classes are denoted $[(X,A),(Y,B)]$, or $[X,Y]$ if $A = B = \phi$. Usually we will consider base pointed classes.

Most problems in algebraic topology are associated with homotopies. Two problems are:

(1) <u>Extension problem</u>: given $A \subset X$ and $g: A \to Y$, does there exist an <u>extension</u> $f: X \to Y$ of g, i.e. a map f which fills in the dotted line of the diagram and makes it commutative?

$$\begin{tikzcd} X \arrow[dr, dotted, "f"] & \\ A \arrow[u, phantom, "\cup"] \arrow[r, "g"] & Y \end{tikzcd}$$

(2) <u>Lifting problem</u> Given a map $\pi: X \to A$, and $g: Y \to A$, can g be <u>lifted</u> to $f: Y \to X$ so that the diagram commutes?

$$\begin{tikzcd} & X \arrow[d, "\pi"] \\ Y \arrow[ur, dotted, "f"] \arrow[r, "g"] & A \end{tikzcd}$$

It was an important discovery that for a large class of spaces (which we shall define later) the answer depends only on the homotopy class of g.

In case 1, we say a pair (X, A) has the <u>homotopy</u> <u>extension</u> <u>property</u> for Y (HEP) iff any map $X \times 0 \cup A \times I \xrightarrow{F_0} Y$ can be extended to a map $X \times I \xrightarrow{F} Y$. In particular, if there is a <u>retraction</u> $r: X \times I \to X \times 0 \cup A \times I$ then $F_0 r$ is such an extension. If (X, A) has the HEP for Y, then the answer to (1) depends on the homotopy class alone of g.

In case (2), we say π has the <u>homotopy</u> <u>lifting</u> <u>property</u> if for any maps $f: Y \to X$ and $F_0: Y \times I \to A$ such that $F_0(y, 0) = \pi f(y)$ there is an extension $F: Y \times I \to X$ such that $\pi F = F_0$. If $\pi: X \to A$ has this property for all Y, it is called a <u>fiber</u> <u>space</u> <u>projection</u>, or X is a <u>fiber</u> <u>space</u> over A.

Examples of both will appear later.

If $f: (X, x_0) \to (Y, y_0)$ and $g: (Y, y_0) \to (X, x_0)$ satisfy $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$, then we say X, Y are <u>homotopically</u> <u>equivalent</u> <u>spaces</u>; or have the same <u>homotopy</u> <u>type</u>. If $\simeq$ were replaced by $=$, f, g would be inverses of one another; this is a weaker equivalence. This is again an equivalence relation

$\mathcal{T}op$ - the category of topological spaces and continuous maps, and $\mathcal{T}op_0$ - for spaces with base point.

$\mathcal{T}op^2$ - the category of pairs of spaces (X,A) where $A \subset X$ (with base points) and maps $f: X \to Y$ such that $f(A) \subset B$, for $f: (X,A) \to (Y,B)$

$\mathcal{G}p$ - the category of groups & homomorphisms

$\mathcal{A}\mathcal{G}p$ - the category of abelian groups & homomorphisms.

<u>THM</u> For a (pointed) space Y, the functions $[\ ,Y] \text{"}{=}T^Y$ and $[Y,\ ]\text{"}T_Y$, defined on spaces by $X \mapsto [X,Y]$ and $X \mapsto [Y,X]$ respectively, and on morphisms by $T^Y(f)[g] = [gf]$ and $T_Y(f)[g] = [fg]$ define a contravariant and a covariant functor, respectively, from $\mathcal{T}op_0$ to $\mathcal{S}_0$.

Functors preserve diagrams and commutativity, so they may be used to give negative answers to problems such as the extension and lifting problems by showing their solution would provide a solution to a problem of similar type in some algebraic category where one can show there is no solution. There will soon be non-trivial examples.

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Now consider the functor on pointed spaces $(X, x_0) \mapsto [(S^1, 1), (X, x_0)] = \pi_1(X, x_0)$. It is sometimes convenient, and clearly equivalent, to use $(I, \{0,1\})$ instead of $(S^1, 1)$. Now $\pi_1(X, x_0)$ admits a group structure, arising from the composition of paths: if $\beta, \alpha : (I, \{0,1\}) \to (X, x_0)$, then $\alpha * \beta$ is defined by $(\alpha * \beta)(t) = \left\{ \begin{array}{ll} \alpha(2t) & \text{for } 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \le t \le 1 \end{array} \right\}$

Math 345 - Algebraic Topology

1. Suppose $f: X_1 \to X_2$ is a map of covering spaces over the connected space B, i.e. the diagram at right is commutative. Suppose that over base points $x_1 \in X_1$ and $x_2 = f(x_1) \in X_2$, f is 1-1 (mapping $p_1^{-1}(b_0) \to p_2^{-1}(b_0)$, where $b_0 = p_1(x_1) = p_2(x_2)$). Show that f is a homeomorphism.

$$\begin{tikzcd} X_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & X_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

2. Suppose X is a covering space over $B \times I$. Let $X_0 = X|B\times 0$

Then there is a commutative diagram, as at right. Show this — you can use the relation of the fundamental groups. Deduce that $X \cong X_0 \times I$.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "p"'] & X_0 \arrow[d, "p"] \\ B\times I \arrow[r, "\text{proj.}"'] & B\times 0 \end{tikzcd}$$

3. Obtain the result above directly by using the HLP to show that there is a diagram as at right.

$$\begin{tikzcd} X_0\times I \arrow[r, "\cong"] \arrow[d] & X \arrow[d] \\ B\times I \arrow[r, equal] & B\times I \end{tikzcd}$$

* The only Compact 1-manifold is S^1

4. Assume #2,3 are true — show that they imply the HLP. You can use the notion of induced bundle.

~~5.~~ Show that if $f, g : A \to B$ and $f \simeq g$, and if $X \xrightarrow{P} B$ is a covering space, then $f^*X \cong g^*X$.

6. Suppose B is obtained from a compact manifold M of dimension 2 by deleting a finite number of points. Suppose $\mathbb{X} \xrightarrow{P} B$ is a covering with finitely many sheets. Show how to embed $\mathbb{X}$ in a compact manifold $\hat{\mathbb{X}}$ so that p can be extended to a map $\hat{\mathbb{X}} \xrightarrow{\hat{P}} M$ (not a covering in general). Hint: each deleted point has a deleted nbhd $\cong S^1 \times I$.

7. Same situation as #6, except dimension of $M > 2$. Show then that $\hat{p} : \hat{\mathbb{X}} \to M$ will be a projection of a covering.

8. Suppose $\mathbb{X} \xrightarrow{P} M$ is a covering space and M is a differential manifold. Show then that there is a differential structure on $\mathbb{X}$ and p is a local diffeomorphism.

MATH 345 January 28, 1970
Algebraic Topology

1. Suppose $y_0, y_1 \in Y$, arcwise connected. Explain how $\pi(Y,y_0)$ and $\pi(Y,y_1)$ are related.

2. State the homotopy lifting property for a covering space $p : \tilde{X} \to X$ and a space Z. Show how this implies $p_{\#}$ is a monomorphism.

3. Suppose $p : \tilde{X} \to X$ is a covering projection (all locally arcwise connected and connected). Suppose $\pi(X,x_0) \cong$ group of permutations of $\{1,2,3\}$ and suppose $p_{\#}\pi(\tilde{X},\tilde{x}_0) = \{id,(12)\}$ while $f : (Z,z_0) \to (X,x_0)$ and $f_{\#}\pi(Z,z_0) = \{id,(23)\}$.

(a) Is there a lift $\tilde{f}$ of f such that $\tilde{f}(z_0) = \tilde{x}_0$?

(b) Is there a lift $\tilde{f}$ of f, with no restriction on $\tilde{f}(z_0)$?

4. Show that the covering space of Problem 3 could not be obtained from $\tilde{X} = RP_n$ by taking the orbit space of an action of Z_3 for X. Could X be a topological group?

$G \quad \pi(X)$

5. (a) If $(Z, A \cup B)$ has the homotopy extension property for Y, show that $(Z/B, A/A \cap B)$ also does (A,B closed).

(b) Show that if $z_0 \in Z$ has a contractible closed nbhd, then (Z,z_0) has the absolute HEP (i.e., for all Y).

6. Show that $R^2 - \{0\}$ is homotopy equivalent to S^1. Show that this implies $\pi(S^1) \cong \pi(R^2-\{0\})$ and that this holds for any covariant homotopy functor from topological spaces and continuous maps to groups and homomorphisms, $X \to \pi(X)$, $f \to \pi(f)$.

$\dfrac{rt}{\sqrt{rt^2+1}}$ $\quad f(0) = 1 \quad$ $rt+1$

$f(1) = \cancel{1} \quad t^2 + rt - 1 \quad (r+1)t$

$rt \quad \text{if } r > 1$

$rt + r(t-1)^0$

$r + (1-t)1-(1-r$

#5

Algebraic Topology 3/23/70

(1) Suppose ξ is a trivial bundle over B, i.e. isomorphic to the product bundle. Show that there is a bundle map from ξ to the bundle consisting of a single fiber. Do this in detail from def's, paying attention to the group.

well?

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(2) Show that the induced bundle of any trivial bundle is trivial. Do this from the defn's, and from problem (1).

(3) Show that every line bundle (group $GL(1) = \{(1), (-1)\}$) over S^1 is isomorphic to the product bundle or the Möbius band.

(4) How would you modify the arguement of (3) to apply to any vector bundle over S^1?

(5). Show that for any compact manifold without boundary the identity map is not homotopic to a map which reverses orientation. Show that the opposite holds for R^n.

(6) Describe a map $S^n \to S^n$ of degree 2.

7. In this problem we shall use the notation $x = (x_1, x_2)$ for a point in R^2, so $\frac{\partial}{\partial x_1} = (1,0)$ and $\frac{\partial}{\partial x_2} = (0,1)$. Consider the map: $\phi : R^2 - \{0\} \to R^2 - \{0\}$ defined by $\phi(x) = x/(|x|^2)$. Compute the matrix of $d\phi(x)$, i.e., $(a_i{}^j)$ where $d\phi(x)(\frac{\partial}{\partial x_i}) = \Sigma a_i{}^j \frac{\partial}{\partial x_j}$.

8. If one restricts $d\phi$ (of Problem 7) to S^1 $(|x| = 1)$ then there is defined $\Phi : S^1 \to R^2 - \{0\}$ by

$$\Phi(x) = d\phi(x)(\tfrac{\partial}{\partial x_1})$$

Now $\pi(R^2 - \{0\}) \cong \pi(S^1) \cong Z$. What integer is associated to $[\Phi]$?

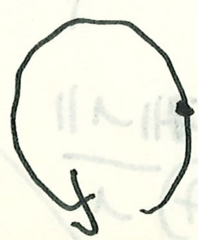
HINT: Write Φ in co-ordinates. Using complex numbers, can you represent Φ as $z \to z^n$?

9. Let $R_0 = R^2 - \{0\}$ and $T_0 = R^2 \times R_0 \cup_\phi R^2 \times R_0$, i.e., the quotient space of the disjoint union $R^2 \times R_0 \cup R^2 \times R_0$ in which (x,X) is identified with $(\phi(x), d\phi(x)(X))$, for $x \in R_0$. Using Van Kampen's theorem and problems 6,8 determine $\pi(T_0)$.

10. Suppose $\gamma : R^4 \to R^4$ $(\cong C^2)$, and in complex coordinates $\gamma(z_1, z_2) = (e(\frac{\pi i}{2})z_1, e(\frac{2\pi i}{3})z_2)$. Then γ is orthogonal, so $\gamma : S^3 \to S^3$. Let L = orbit space of S^3 under the group generated by γ. Is $S^3 \to L$ a covering? What is $\pi(L)$?

$\cos \frac{\pi}{2} z_i + i \sin \frac{\pi}{2} z_1$

4×3



Algebraic Topology 3/13/70

Fri

Pre Work #4

(1.) Show that the quotient map $(z_0, \ldots, z_n) \mapsto [z_0, \ldots, z_n]$, defined
5 for $|z_0|^2 + \cdots + |z_n|^2 = 1$, defines a bundle projection $S^{2n+1} \to \mathbb{C}P_n$
with fiber $= S^1$, group $= S^1$.

(2) Show that any bundle over a contractible (paracompact)
5 space is isomorphic to the trivial bundle.

? More on this

(3) Suppose $A \times F \to A$ is the trivial bundle over A. Show that there is a 1-1 correspondence between sections of the bundle and maps $A \to F$.

(4) As an example of (3.), show that a section of the trivial
5 bundle defined over $B \subset A$ extends to a section over A iff
the corresponding map $B \to F$ extends to $A \to F$.

(5). Now suppose $Y = X \cup_f D_n$ is an adjunction space, i.e. is the
5 quotient of the disjoint union of X and D^n in which $x \in S^{n-1} \subset D^n$
is identified with $f(x) \in X$. Suppose ξ is a bundle over Y which has a section over X, say g. Let $j: D^n \to Y$ be the obvious map. By considering $j^*\xi$, show how to associate to ξ, g a map $O(g): S^{n-1} \to F_\xi$ such that $O(g) \simeq$ constant iff g extends over Y (F_ξ are conn.)

Algebraic Topology. 4/15/70

(1.) Suppose G is a Lie group which acts on X transitively, so for every $x, y \in X$ there is a $g \in G$ and $g(x) = y$. If $x_0 \in X$, we can define $p: G \longrightarrow X$ by $g \longmapsto g(x_0)$ — so p depends on x_0. Define $G_{x_0} \subset G$ to be $\{g \mid g x_0 = x_0\}$. Show p induces a 1-1 continuous map $p': G/G_{x_0} \longrightarrow X$.

(2) With notation of (1), we say p has a local section at $y \in X$ if it locally has a right inverse, i.e. there is a nbhd U of y and a map $f: U \to G$ such that $p \circ f = id$. Show that if p has a local section at one point, it does at every point.

(3) Let $G = SO(n)$ = special orthogonal group on R^n, which acts on $X = S^{n-1}$. Let $x_0 = e_1 = (1, 0 \ldots 0)$, and show that p in this case does have local sections.

(4) In the notation of (1), (2), show that if p does have local sections, then G is a fiber bundle over X with fiber G_{x_0} and group G_{x_0}.

Algebraic Topology - May 14, 1970

1. Show that $\pi_k(D^n, S^{n-1}) \cong \pi_{k-1}(S^{n-1})$. If $D^n_- \subset S^n$ is a hemisphere, show that $\pi_k(S^n, D^n_-) \cong \pi_k S^n$. Use this to show that the homotopy groups π_k do not satisfy the excision axiom of homology theory.

2. Let RP_n = real projective n-space $= S^n/Z_2$. Show that $RP_n = RP_{n-1} \cup_{f_n} e^n$, i.e. RP_n is RP_{n-1} with one n-cell attached by some map f_n. Thus RP_n is a CW complex with one cell in each dimension, and k-skeleton RP_k.

3. Consider the map $RP_n \to RP_n/RP_{n-1} = e^n/\partial e^n = S^n$. Determine the degree of the map $f_{n+1}: \partial e^{n+1} \to RP_n \to RP_n/RP_{n-1} = S^n$.

   n even $\quad$ deg $= \pm 2$
   odd $\quad$ deg $= 0$

4. A sphere S^n can be represented as a complex $e^0 \cup e^n$, so D^{n+1} can be represented as $e^0 \cup e^n \cup e^{n+1}$ with obvious attaching maps. Then let $n+m=N$, $X = D^n \times D^m$. One can describe X as a complex with $X^k = \bigcup_{i+j \le k} e^i \times e^j$. Then $X^N/X^{N-1} \sim S^N$ represented by $D^n \times D^m$, and $X^{N-1}/X^{N-2} \sim S_1^{N-1} \vee S_2^{N-1}$; S_1^{N-1} coming from $S^{n-1} \times D^m$, S_2^{N-1} from $D^n \times S^{m-1}$. Determine the map $\partial D^N \to S_1^{N-1} \vee S_2^{N-1}$. For orientations, observe for $(x,y) \in S^{n-1} \times S^{m-1}$, $v_1 \cdots v_n \cdots v_{n+m} \to v_{[illegible]}\, v_n \cdots v_{n+m}$, & $v_1 \cdots v_{n+m} \to v_1 \cdots v_n v_{n+2} \cdots v_{n+m}$

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Algebraic Topology 3/18/70

Week Fri 3/27/70

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1. Suppose ξ is the trivial bundle over D^n with group G and fiber F, so $E_\xi = D^n \times F$. Suppose a map $\varphi : \xi | S^{n-1} \to \xi | S^{n-1}$ over id is given — this defines a map $\varphi_0 : S^{n-1} \to G$ by

$$\varphi(x, f) = (x, \varphi_0(x) f)$$

if φ is a bundle map. Show that φ extends to a bundle map $\xi \to \xi$ over the identity <u>iff</u> φ_0 is null homotopic, i.e. extends over D^n.

$j^{-1} \circ j'^{-1} = X_\xi$ $(j'^{-1} \circ X_\xi)^{-1} = X_\xi^{-1} \circ j$

Add to this

2. Let $S^n = D^n_+ \cup D^n_-$ be the decomposition into hemispheres, so $D^n_+ \cap D^n_- = S^{n-1}$. Suppose ξ is a bundle over S^n with fiber F and group G. Then there are trivialisations

$$j_+ : \xi | D^n_+ \cong D^n_+ \times F \qquad j_- : \xi | D^n_- \cong D^n_- \times F.$$

Now $j_- \circ j_+^{-1} | S^{n-1} \times F$ is a bundle map $S^{n-1} \times F \to S^{n-1} \times F$ — in notation of (1), define $X_\xi = (j_- \circ j_+^{-1} | S^{n-1} \times F)_0$. Show that if j_+ extends to $\xi \cong S^n \times F$, then X_ξ extends over D^n & conversely. Show that the homotopy class of X_ξ does not depend on j_+, j_-, G connect

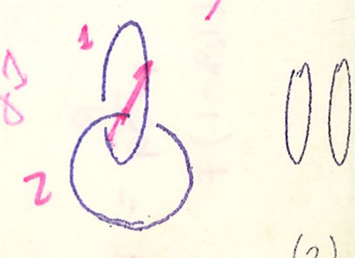
3. Suppose ξ, ψ are bundles over S^n with fiber F and group G. Show that if there is a bundle map $\xi \to \psi$ over the identity, then $\chi_\xi \simeq \chi_\psi$ as maps $S^{n-1} \to G$ (G connected).

4. Show that for G connected, there is a 1-1 correspondence between isomorphism classes of bundles over S^n and $\pi_{n-1}(G, e)$, the set of homotopy classes of maps $S^{n-1} \to G$; in particular the bundles over S^2 with group $GL(2)$ correspond to the integers.

5. Show that there is a homotopy equivalence between $GL(n, R)$ and $O(n)$. Give explicit references for the linear algebra used and explain how it is used — e.g. Halmos Finite dimensional Vector spaces, §83.

6. A (two) <u>link</u> in R^3 is a map of the disjoint union of two copies $S_1 \cup S_2$ of the circle S^1 which is an embedding — so each S_i is embedded and they don't meet. Let f_i be the map on S_i.
An isotopy of links is a map $(S_1 \cup S_2) \times I \to R^3$ which for each t gives a link $(S_1 \cup S_2) \times t \to R^3$. Use the map $S_1 \times S_2 \to S^2$ by $(x, y) \to \dfrac{f_1(x) - f_2(y)}{|f_1(x) - f_2(y)|}$ to show (1) is <u>not</u> isotopic to (2).

deg 1



(1) (2)

Math 345 Assignment 2 Due Oct 28, 1970

1. Let $p: X \to Y$ be an identification, $f: X \to Z$ a closed map and $g: Y \to Z$ s.t. $gp = f$. Show that g is closed.

2. Let Z be a Hausdorff space, Y closed in Z, $f: E^n \to Z$ s.t. $f(S^{n-1}) \subset Y$, $f(E^n - S^{n-1}) \subset Z - Y$ and $f|_{E^n - S^{n-1}}$ is a homeomorphism onto $Z - Y$. Show that $Z \simeq E^n \cup_{f|_{S^{n-1}}} Y$. (Hint: construct a map $E^n \cup_{f|_{S^{n-1}}} Y \to Z$ and note f and inclusion are closed. Use prob 1) (This is 19.4 in text)

3. Prove $P^n = E^n \cup_f P^{n-1}$ where $f: S^{n-1} \to P^{n-1} = S^{n-1}/R$ is the canonical map. (Hint: See 19.10 in text)

4. A subset $A \subset R^n$ is called convex if given $x_0, x_1 \in A$, $tx_0 + (1-t)x_1 \in A$ for $0 \le t \le 1$. Prove that every convex subset of R^n is contractible.

5. Let X, Y, Z, W be spaces, $f: Y \to Z$, $g: Z \to W$ be maps. Define $f_*: [X, Y] \to [X, Z]$ by $f_*([x]) = [fx]$. Show $g_* f_* = (gf)_*$, $(1_Y)_* = 1_{[X,Y]}$, if $f_1 \simeq f_2$, $f_{1*} = f_{2*}$ and if f is a homotopy equivalence, f_* is a bijection.

6. Let $p_1: X \times Y \to X$, $p_2: X \times Y \to Y$ be the projections. Prove that given $[f] \in [Z, X]$, $[g] \in [Z, Y]$ $\exists$ a unique $[h] \in [Z, X \times Y]$ s.t. $p_{1*}([h]) = [f]$, $p_{2*}([h]) = [g]$.

7. Using 6 prove that p_1 is a homotopy equivalence if and only if Y is contractible.

8. Let X, Y be path connected. Prove $X \times Y$ is.

9. Let X be a space, $\{U_\alpha\}_{\alpha \in A}$ a collection of path connected subsets such that $\bigcap_{\alpha \in A} U_\alpha \neq \emptyset$. Show that $\bigcup_{\alpha \in A} U_\alpha$ is path connected.

Assignment # 3
Due Nov. 20

Math 345

Nov 6, 1970

1. Let $x_0 \in X$, $P \subset X$ be the path component of x_0, $i : P \to X$ the inclusion. Show that $\pi(i)$ is an isomorphism $\pi(P, x_0) \to \pi(X, x_0)$

2. Let X be path connected, SX be the suspension of X. Prove that SX is simply connected.

3. Let X be a space, $x_0 \in X$, $p \in S^n$. Let $A = \{f : S^n \to X \mid f(p) = x_0\}$ and $\bar{A}$ be the set of homotopy classes of maps in A rel $\{p\}$. Let $B = \{g : I^n \to X \mid g(x) = x_0$ for x on the boundary of $I^n = \partial I^n\}$ and $\bar{B}$ be the set of homotopy classes of maps in B rel ∂I^n. Show that there is a 1-1 correspondance between $\bar{A}$ and $\bar{B}$. (Hint: let $S^n = I^n/\partial I^n$ since $S^n \approx E^n/S^{n-1}$, $E^n \approx I^n$, $\partial I^n \approx S^{n-1}$)

4. Show that every map $f : S^n \to S^1$ for $n \geq 2$ such that $f(p) = 1$ is homotopic to the constant map at 1 rel $\{p\}$ (Hint: Consider the appropriate map $I^n \to S^1$ and factor it through $ex : \mathbb{R}^1 \to S^1$. Why does this not work for $n = 1$?)

5. Let G, H be arbitrary Groups, φ a group homomorphism. Show $\exists$ a unique $\varphi' : G/[G,G] \to H/[H,H]$ s.t. φ' is a homomorphism and

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\eta_G"] & H \arrow[d, "\eta_H"] \\ G/[G,G] \arrow[r, "\varphi'"] & H/[H,H] \end{tikzcd}$$

where η_G, η_H are the cannonical maps.

6. For any path connected space X define $H(X, x_0) = \dfrac{\pi(X, x_0)}{[\pi(X,x_0), \pi(X,x_0)]}$. Let α be a path from x_0 to x_1 and define $\alpha'_* : H(X, x_0) \to H(X, x_1)$ to be the map obtained from α_* by problem 5. Show that α'_* does not depend on α, i.e. if β is a path from x_0 to x_1 $\beta'_* = \alpha'_*$. Show also that α'_* is an isomorphism.

7. In view of problem 6 if X is path connected we can define $H(X) = H(X, x_0)$ where x_0 is any point in X. If Y is path connected and $f : X \to Y$ is a map show $\exists$ a well defined homomorphism $H(f) : H(X) \to H(Y)$ which depends only on the homotopy class of f.

M345 Assignment 4 page 1 Due Dec 18, 1970

1. Let R be a ring, P an R-module. Prove the following are equivalent:

 i) There is a R-module M s.t. $P \oplus M$ is free

 ii) Every exact sequence of R-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$$ splits

 iii) Given a surjective R-linear map $g: A \longrightarrow B$ and a R-linear map $f: P \longrightarrow B$ $\exists$ an R-linear map $h: P \longrightarrow A$ s.t. $gh = f$.

(Hint: prove i $\Rightarrow$ iii $\Rightarrow$ ii $\Rightarrow$ i) If P satisfies any of these conditions it is called projective.

2. Prove that every f.g. projective $\mathbb{Z}$-module is free. Let $R = \mathbb{Z} \times \mathbb{Z}$ (pointwise addition & multiplication) Let $I = \{(z,0) \in \mathbb{Z}\times\mathbb{Z}\}$. Show I is an ideal of R, hence an R-module, I is projective but I is not free.

3. Let D be an R-module, $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ an exact sequence of R-modules. Show

$$0 \longrightarrow \mathrm{Hom}(D,A) \xrightarrow{f_*} \mathrm{Hom}(D,B) \xrightarrow{g_*} \mathrm{Hom}(D,C)$$

is exact where $\mathrm{Hom}(D,A) = R\text{-mod}[D,A]$ as a R-module and f_*, g_* the induced homomorphism.

If D is free show

$$0 \longrightarrow \mathrm{Hom}(D,A) \xrightarrow{f_*} \mathrm{Hom}(D,B) \xrightarrow{g_*} \mathrm{Hom}(D,C) \longrightarrow 0$$

is exact.

4. A category G such that ob G is a set and each morphism of G is an isomorphism is called a groupoid. Show that for $X \in$ ob G, $G[X,X]$ is a group. Show that there is a functor $F_G : G \longrightarrow Gp$ (category of groups) s.t. $F_G(X) = G(X,X)$ and $F_G(\alpha)(f) = \alpha f \alpha^{-1}$ for $\alpha \in G[X,Y]$, $f \in G[X,X]$.

5. Briefly sketch a proof that $\mathcal{P}(X)$ - X a topological space - given by ob $\mathcal{P}(X) = X$, $\mathcal{P}(X)[x_0, x_1]$ = set of homotopy classes of maps $\gamma : I \longrightarrow X$ s.t. $\gamma(0) = x_0$, $\gamma(1) = x_1$ is a groupoid. Note $\pi(X, x_0) = F_{\mathcal{P}(X)}(x_0)$. Given a map $f : X \longrightarrow Y$ construct a functor $\mathcal{P}(f) : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$ and a natural transformation $\eta : F_{\mathcal{P}(X)} \longrightarrow F_{\mathcal{P}(Y)} \mathcal{P}(f)$ s.t. $\eta(x_0) = \pi(f) : \pi(X, x_0) \longrightarrow \pi(Y, f(x_0))$.

6. Let K be a category, $X, Y \in$ ob K, $h_X, h_Y : K \longrightarrow Ens$ as defined in class. Let $\eta : h_X \longrightarrow h_Y$ be a natural transformation. Show $\exists !$ $\alpha \in K[Y,X]$ s.t. $\eta(Z)(\gamma) = \gamma\alpha$ $\forall$ $Z \in$ ob K, $\gamma \in h_X(Z)$. (Hint: if $\gamma \in h_X(Z)$, $\gamma = h_X(\gamma)(1_X)$.)

7. A functor $F : K \longrightarrow Ens$ is called representable if $\exists$ a natural isomorphism $\eta : h_X \longrightarrow F$. In this case we say F is represented by X. Prove that if F is represented by X and Y then X and Y are isomorphic. (Hint: use 6)

8. Let $F_X : R\text{-mod} \longrightarrow Ens$ be given by $F_X(A) = \{\alpha : X \longrightarrow A \mid f \text{ is a function}\}$ where X is some fixed set, $F(f)(\alpha) = f\alpha$. Show that F_X is represented by the free R-module on X.

9. Let X be a topological space, $A \subset X$ a closed subspace. Define $F : Top^* \longrightarrow Ens$ by $F(Z, z_0) = \{f : X \longrightarrow Z \mid f \text{ is a map}, f(A) = \{x_0\}\}$, $F(f)(\alpha) = f\alpha$. Show F is represented by $(X/A, [A])$.

10. Let A, B be R-modules. Define $F : R\text{-mod} \longrightarrow Ens$ by $F(C) = \{(f,g) \mid f : A \longrightarrow C,\ g : B \longrightarrow C \text{ are } R\text{-linear}\}$, $F(h)(f,g) = (hf, hg)$. Show F is representable and find its representing object in R-mod.

Math 345 Due Jan 6, 1971

Assignment 5

"Have a diagramatic Christmas"

1. (Nine Lemma) Let

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
(1)\quad 0 \arrow[r] & A_1 \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & 0 \\
(2)\quad 0 \arrow[r] & A_2 \arrow[r, "\kappa"] \arrow[d] & B_2 \arrow[r, "\sigma"] \arrow[d] & C_2 \arrow[r] \arrow[d] & 0 \\
(3)\quad 0 \arrow[r] & A_3 \arrow[r] \arrow[d] & B_3 \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

be a commutative diagram of R-modules where the columns are exact.

Prove:

i) If rows (1), (2) are exact so is (3)

ii) If rows (2), (3) are exact so is (1)

iii) If rows (1), (3) are exact and $\sigma\kappa = 0$ (2) is exact.

(Hint: Consider the rows as chain complexes & use exact homology sequence. If you prefer you may chase the diagram but I would hope that you don't).

2. (Five Lemma) Let the following be a commutative diagram of R-modules

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "\alpha_1"] & A_2 \arrow[r] \arrow[d, "\alpha_2"] & A_3 \arrow[r] \arrow[d, "\alpha_3"] & A_4 \arrow[r] \arrow[d, "\alpha_4"] & A_5 \arrow[d, "\alpha_5"] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

and assume that the rows are exact. If $\alpha_1, \alpha_2, \alpha_4, \alpha_5$ are isomorphisms show that α_3 is also an isomorphism. (I'm afraid you will have to chase this one.)

3. Let $A \xrightarrow{f} B$ be R-linear. Define the co-kernel of f

$\operatorname{coker} f = B/\operatorname{Im} f$. Prove for any two R-linear maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

That there is an exact sequence

$$0 \to \ker f \to \ker gf \to \ker g \to \operatorname{coker} f \to \operatorname{coker} gf \to \operatorname{coker} g \to 0$$

4. Let $f : C = \{C_n, \partial_n\} \longrightarrow K = \{K_n, d_n\}$ be a chain transformation. Define $M(f) = \{M_n, \delta_n\}$ by $M_n = C_{n-1} \oplus K_n$, $\delta_n(c,k) = (-\partial c,\ dk + f_{n-1}(c))$ Show $M(f)$ is a chain complex, (It is called the mapping cone). Construct a chain transformation $i : K \longrightarrow M(f)$ by $i_n(k) = (0,k)$. Let $C^+ = \{C_n^+, \partial_n^+\}$ where $C_n^+ = C_{n-1}$ $\partial_n^+ = -\partial_{n-1}$. Define $\pi : M(f) \longrightarrow C^+$ by $\pi_n(c,k) = c$, and show π is a chain transformation. Note $H_{n-1}(C) = H_n(C^+)$.

5. Using the exact homology sequence show

$$\cdots \longrightarrow H_n(K) \xrightarrow{H_n(i)} H_n(M(f)) \xrightarrow{H_n(\pi)} H_{n-1}(C) \xrightarrow{H_n(f)} H_{n-1}(K) \longrightarrow \cdots$$

is exact. (Hint: show $H_n(f)$ is the connecting homomorphism in the exact homology sequence).

6. Given a chain homotopy $s_n : f \simeq g : C \longrightarrow K$ construct a chain map $h : M(f) \longrightarrow M(g)$ by

$$h(c,k) = (c,\ k + s_{n-1}(c))$$

Show $H_n(h)$ is an isomorphism.

(Hint: show

$$\begin{tikzcd}
H_n(C) \arrow[r] \arrow[d, "1"] & H_n(K) \arrow[r] \arrow[d, "1"] & H_n(M(f)) \arrow[r] \arrow[d, "H_n(h)"] & H_{n-1}(C) \arrow[r] \arrow[d, "1"] & H_{n-1}(K) \arrow[d, "1"] \\
H_n(C) \arrow[r] & H_n(K) \arrow[r] & H_n(M(g)) \arrow[r] & H_{n-1}(C) \arrow[r] & H_{n-1}(K)
\end{tikzcd}$$

commutes and use 5-lemma)

Reference for this page: Mac Lane, Homology pp 46 - 49.