Functional Analysis Summer 2016

Bellenot

March 16, 2016

Introduction

The word functional in Functional Analysis means several different things. First, the most important vector spaces considered are spaces of functions. There are many function spaces of interest. Examples include a space of continuous functions and the Hilbert space of functions f with $\int |f|^2\, dx < \infty$.

In another sense, a functional is an operator or map from a vector space to the scalar field. For example, π_2 on $\mathbb{R}^3$ defined by $\pi_2(\langle x_1, x_2, x_3\rangle) = x_2$ is a functional. It is possible to try to push questions on the big space to the question scalar field by the use of the collection of certain functionals. Such properties often have the word "weak" as an adjective. Many important objects are functionals. Energy is a functional, integration is a functional, evaluation at a point is a functional.

1 An example

Eventually this section is about solutions to the IVT

$$\phi'(t) = f(t, \phi((t))$$

$$\phi(0) = y_0$$

but consider the operator

$$\phi(t) = T\psi(t)$$

$$\phi(x) = y_0 + \int_0^x f(t, \psi(t))\, dt$$

which takes functions $\psi(x)$ to functions $\phi(x)$. If f and ψ are continuous, the integral exists. The fundemental theorem of calculus says

$$\phi'(t) = f(t, \psi(t))$$

and clearly $\phi(0) = y_0$. If we could find a function ϕ so that $\phi = T\phi$ it would solve the IVP.

Instead of working with these particulars, functional analysis abstracts by finding a fix point of a nice operator on a metric space. In this case, there is some $\tau > 0$ so that the space is the collection of continuous functions, $C([0, \tau])$, with the sup norm

$$\|f(t)\| = \sup\{|f(t)| : t \in [0, \tau]\} = \max\{|f(t)| : t \in [0, \tau]\}.$$

The abstract result used is the contraction mapping principle that says on a complete metric, a contraction map has a unique fixed point. In this case a contraction map means there is a $c < 1$ so that for all x and y

$$\|Tx - Ty\| < c\|x - y\|.$$

2 Grading

This course has an effort based minimal grade. If you show up, attend all the classes, and take all the quizzes and the test you will not get below a B+ in the class. There are also problem sets for higher grades.