Truth with respect to an ultrafilter

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A free ultrafilter $\mathscr{U}$

Let I be an infinite index set, in the examples $I = \mathbb{N}$ but often I is a huge set. Let $\mathscr{U}$ be a free ultrafilter on I, this means the following conditions are true

1. $\emptyset \notin \mathscr{U}$
2. $A \in \mathscr{U}$ and $A \subseteq B \Rightarrow B \in \mathscr{U}$
3. $A, B \in \mathscr{U} \Rightarrow A \cap B \in \mathscr{U}$
   This is related to (FIP): $A, B \in \mathscr{U} \Rightarrow A \cap B \neq \emptyset$
4. (free) $\cap \mathscr{U} = \emptyset$
5. (ultra) $A \subseteq I$, then either $A \in \mathscr{U}$ or $I \backslash A \in \mathscr{U}$

Define an equivalence relation on $(I \to \mathbb{N})$ by

$$f \equiv g \iff \{x \in I | f(x) = g(x)\} \in \mathscr{U}$$

the relations is obviously symmetric, it is reflexive since $I \in \mathscr{U}$ and transitive by the FIP. Note that on the collection of equivalence classes the definitions below are well defined

$$f < g \iff \{x \in I | f(x) < g(x)\} \in \mathscr{U}$$

$$f \leq g \iff \{x \in I | f(x) \leq g(x)\} \in \mathscr{U}$$

$$f \equiv g \Rightarrow f + 1 \equiv g + 1$$

The ultra property gives that this $\leq$ ordering is a linear ordering as one of the sets $\{x|f(x) < g(x)\}, \{x|f(x) = g(x)\}$, or $\{x|f(x) > g(x)\}$ must belong to the ultrafilter. We will the space $(I \to \mathbb{N})/\equiv$ as ${}^*\mathbb{N}$

Infinitely large elements

Let $I = \mathbb{N}$. We can think of $\mathbb{N} \subseteq {}^*\ltimes$ by identifying $n \in \mathbb{N}$ with the function that is constantly n. Consider $f(i) = i$, this is and infinitely large element in the sense $n < f$ for all $n \in \mathbb{N}$. Indeed the set $\{x|f(x) > n\} = \{n+1, n+2, \dots\} \in \mathscr{U}$ since the ultrafilter is free. In particular, ${}^*\mathbb{N}$ is strictly bigger than $\mathbb{N}$

The successor function

Peano axioms for $\mathbb{N}$ require as successor function S, on ${}^*\mathbb{N}$ we define $S(f)$ to be $g = f + 1$. That is $g(i) = f(i) + 1$ for all $i \in I$. Lets check Peano's axioms for ${}^*\mathbb{N}$

1. ${}^*\mathbb{N}$ is a set which contains the element $0, 0 \in {}^*\mathbb{N}$.
2. S is a function on ${}^*\mathbb{N}, S : {}^*\mathbb{N} \to {}^*\mathbb{N}$

3. S is an injection, $Sf = Sg \Rightarrow f = g$.

   In ${}^*\mathbb{N}$ this means $\{i|f(i)+1 = g(i)+1\} = \{i|f(i) = g(i)\} \in \mathscr{U}$.

4. For each f, $S(f) \neq 0$

   Indeed $\{i|f(i)+1 \neq 0\} = I \in \mathscr{U}$

5. Induction Principle. We have to modify what it means to be a set for this to be true. Since $\mathbb{N} \subset {}^*\mathbb{N}$ would obviously be a contraexample.

Set redefinition one, star sets

Suppose we restrict the word set to be 'stars' of subsets of $\mathbb{N}$ If $A \subseteq \mathbb{N}$, then define

$$ {}^*A = \{f|\{i|f(i) \in A\} \in \mathscr{U}\} $$

If $0 \in {}^*A$ then $0 \in A$ and if $f \in {}^*A \Rightarrow S(f) \in {}^*A$ then if $n \in A \Rightarrow S(n) \in A$ which implies $A = \mathbb{N}$ and hence ${}^*A = {}^*\mathbb{N}$. The these kind of star sets are called standard. All standard sets satisfy the inductive principle.

Set redefinition two, internal sets

An internal set A is one that belongs to ${}^*\mathscr{P}(\mathbb{N})$. What does this mean. Basically there is some statement $P(x)$ so that $A = \{n|P(n)\}$. It turns out that these sets also satisfy the induction principle. Sets that are not internal are called external. The subset $\mathbb{N} \subseteq {}^*\mathbb{N}$ is external. That is all the problem sets are external. An easy example of an internal non-standard set is the $A = \{g|g \geq f\}$ where f is the infinite function defined several sections ago.

We illustrate this with a binary relation R given by $xRy \iff x \leq y$, the for each x, the set $\{y|xRy\}$ is a subset of $\mathbb{N}$. So for each $f : I \to \mathbb{N}$, the set

$$ A = \{g : I \to \mathbb{N}|\{i|f(i) \leq g(i)\} \in \mathscr{U}\} $$

is internal and is, in fact the set A in the paragraph above. This can be generalized to sets discribe by n-ary relations.

The reader may sigh in relieve as we skip this detour into formal logic. But the point is that the subset $\mathbb{N} \subseteq {}^*\mathbb{N}$ cannot be singled out in the language. And hence the inductive principle is valid in $\mathbb{N}$.

Filters in General

A collection of sets $\mathscr{F}$ is a filter if it satisfies the first three conditions of an ultrafiler. It is easy to get a free filter on any infinite set I. Just let

$$ \mathscr{F} = \{X \subseteq I|I \backslash X \text{ is finite}\} $$

It is the collection of co-finite sets. Every free ultrafilter will contain this $\mathscr{F}$. Non-free filters are fixed, each $a \in I$ has a principle ultrafilter $\mathscr{U}_a = \{X \subseteq I|a \in I\}$ every fixed filter is a subset of a principle ultrafilter.

Do free ultrafilters exist? Yes but one needs something like the Axiom of Choice. For example, a proof using Zorn's Lemma is easy since the union of a chain of filters is still a filter. And maximal filters are ultrafilters.

Exercise: prove these statements.

The existence of ultrafilters is strictly weaker then Axiom of Choice as it follows from the weaker Boolean Prime Ideal Theorem. A prime ideal in the boolean algebra of subsets of I is an ultrafilter, that is it is a maximal ideal.

Exercise: prove these cliams.