MAP 3306 eMath2 **Test 3** 3 Aug 2007

Directions: Show **ALL** work for credit; Give **EXACT** answers when possible; Start each problem on a **SEPARATE** page; Use only **ONE** side of each page; Be neat; Leave margins on the left and top for the **STAPLE**; Nothing written on this page will be graded;

1. For each solution below, give the **type of equation** (wave, heat, Laplace) it solves, the **number of space dimensions** (1, 2 3), the **coordinate system used** (rectangle, polar, cylinderical, spherical), and the **region** (square, rectangle, disk, infinite line, sphere)

$$\begin{array}{ll} A. & u(r,\theta) = r^n \sin(n\theta) \\ B. & u(r,\phi) = r^n P_n(\cos\phi) \\ C. & u(r,\theta,t) = J_m(\lambda_{m,n} r)\sin(m\theta)\cos(c\lambda_{m,n} t) \\ D. & u(x,y,t) = \sin(m\pi x)\sin(n\pi y)\cos(c\pi\sqrt{n^2+m^2}\,t) \\ E. & u(x,y,t) = \sin(m\pi x)\sin(n\pi y)\exp(-c^2\pi^2(n^2+m^2)t) \end{array}$$

2. Use Fourier Transforms to solve $u_x + 5u_t = 0$; $u(x,0) = f(x)$

3. Show $u(\rho,\theta,\phi) = 3\rho\cos\phi - 3\rho^3\cos\phi + 5\rho^3\cos^3\phi$ is a solution to

$$u_{\rho\rho} + \frac{2}{\rho}u_\rho + \frac{1}{\rho^2}(u_{\phi\phi} + \cot\phi\, u_\phi) = 0$$

   with the boundary condition

$$u(1,\theta,\phi) = 5\cos^3\phi$$

4. Use the integral definition (3) to find the Fourier Transform of $f(x) = \begin{cases} -5x & |x| < 2 \\ 0 & \text{otherwise} \end{cases}$

   You might need to use (18) and/or (19) to simplify your answer.

5. True or False and a brief reason why or why not.

   (a) $\mathscr{F}[\sqrt{2\pi}e^{-x^2/\sqrt{2}}] = \sqrt{2\pi}e^{-w^2/\sqrt{2}}$

   (b) If $f(x) = x$, then the convolution $f * f = x^3/3$

   (c) The solution $u(x,t)$ to the heat equation on $(-\infty,\infty)$ with initial temperature

$$u(x,0) = f(x) = \begin{cases} 1 & |x| < 2 \\ 0 & \text{otherwise} \end{cases}$$

   has $u(10^{100}, 10^{-100}) > 0$

   (d) The Laplace transform of $u_x(x,t)$ is $\frac{\partial}{\partial x}U(x,s)$ and the Fourier transform of $u_x(x,t)$ is $iw\widehat{u}(w,t)$

   (e) The Fourier transform of $\frac{\sin 2x}{2x}$ is non-zero for all w with $|w| < 4$

   (f) If $f(x) = \sum_{m=1}^\infty \frac{1}{m^2}\sin mx$ and $g(y) = \sum_{n=1}^\infty \frac{1}{n}\sin ny$ then

$$f(x)g(y) = \sum_{m=1}^\infty\sum_{n=1}^\infty \frac{1}{m^2 n}\sin mx \sin ny$$

   (g) The notation $J_m(x)$ is used for the Legendre polynomial of degree m.

   (h) In polar coordinates, using $x = r\cos\theta$ and $y = r\sin\theta$ then

$$u_{r\theta} = -ru_{xx}\cos\theta\sin\theta + ru_{xy}\cos^2\theta - ru_{yx}\sin^2\theta + ru_{yy}\sin\theta\cos\theta$$

   (i) The first three Legendre polynomials are x, $\frac{1}{2}(3x-1)$ and $\frac{1}{2}(5x^3-3x)$.

   (j) The PDE $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = u_{tt}$ is linear.

The Fourier transform formula's are on the other side.

$$\mathscr{F}[f(x)] = \widehat{f}(w) \text{ or simply } \mathscr{F}[f] = \widehat{f} \tag{1}$$

$$\mathscr{F}^{-1}[\widehat{f}(w)] = f(x) \text{ or simply } \mathscr{F}^{-1}[\widehat{f}] = f \tag{2}$$

$$\mathscr{F}[f(x)](w) = \widehat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx \tag{3}$$

$$\mathscr{F}^{-1}[\widehat{f}(w)](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(w) e^{iwx}\, dw \tag{4}$$

$$\mathscr{F}[u(x,t)](w,t) = \widehat{u}(w,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-iwx}\, dx \tag{5}$$

$$\mathscr{F}^{-1}[\widehat{u}(w,t)](x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{u}(w,t) e^{iwx}\, dw \tag{6}$$

$$\mathscr{F}[af(x) + bg(x)](w) = a\widehat{f}(w) + b\widehat{g}(w) \tag{7}$$

$$\mathscr{F}[f'(x)](w) = iw\widehat{f}(w) \tag{8}$$

$$\mathscr{F}[f''(x)](w) = -w^2 \widehat{f}(w) \tag{9}$$

$$\mathscr{F}[\frac{\partial}{\partial x} u(x,t)](w,t) = iw\widehat{u}(w,t) \tag{10}$$

$$\mathscr{F}[\frac{\partial^2}{\partial x^2} u(x,t)](w,t) = -w^2 \widehat{u}(w,t) \tag{11}$$

$$\mathscr{F}[\frac{\partial}{\partial t} u(x,t)](w,t) = \frac{\partial}{\partial t} \widehat{u}(w,t) \tag{12}$$

$$\mathscr{F}[\frac{\partial^2}{\partial t^2} u(x,t)](w,t) = \frac{\partial^2}{\partial t^2} \widehat{u}(w,t) \tag{13}$$

$$[f * g](x) = \int_{-\infty}^{\infty} f(w) g(x-w)\, dw = [g * f](x) = \int_{-\infty}^{\infty} f(x-w) g(w)\, dw \tag{14}$$

$$\mathscr{F}[f * g] = \sqrt{2\pi}\, \widehat{f}\widehat{g} \tag{15}$$

$$f(x-a) = \mathscr{F}^{-1}[e^{-iwa} \widehat{f}(w)] \tag{16}$$

$$\mathscr{F}[\exp(-ax^2)] = \frac{1}{\sqrt{2a}} \exp(\frac{-w^2}{4a}) \tag{17}$$

$$\sin wa = \frac{e^{iwa} - e^{-iwa}}{2i} \tag{18}$$

$$\cos wa = \frac{e^{iwa} + e^{-iwa}}{2} \tag{19}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-iwx}\, dx = \sqrt{\frac{2}{\pi}}\, \frac{\sin aw}{w} \tag{20}$$

$$\mathscr{F}\left[\frac{\sin ax}{x}\right] = \sqrt{\frac{\pi}{2}} \text{ if } |w| < a; 0 \text{ otherwise} \tag{21}$$

$$\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ax} e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}(a+iw)} \tag{22}$$