Section 11.5 Example done by Laplace Transform

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1 The Problem

The problem we are solving is the sawtooth forcing function.

$$y'' + 4y = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

We wanted the general solution.

2 Step by step as done in class

First, we solve the associated homogeneous problem $y'' + 4y = 0$ by looking at the characteristic polynomial $x^2 + 4 = 0$ which has roots $\pm 2i$ and this gives general solution $y = C_1 \cos 2x + C_2 \sin 2x$.

Next we need to solve for the particular solution to $y'' + 4y = (1/n) \sin nx$ for $n = 1, 2, 3, \ldots$. We used the method of undetermined coefficients. For $n \neq 2$, the "guess" would be $y_{\text{particular}} = A \cos nx + B \sin nx$ and for $n = 2$ the guess would be $y_{\text{particular}} = x(A \cos nx + B \sin nx)$ because $2i$ is a root of characteristic polynomial (once).

3 The case $n \neq 2$

Lets plug our guess into our equation

$$\begin{aligned}
y &= A \cos nx + B \sin nx \\
y' &= -An \sin nx + Bn \cos nx \\
y'' &= -An^2 \cos nx - Bn^2 \sin nx \\
4y &= 4A \cos nx + 4B \sin nx \\
\hline
\frac{1}{n} \sin nx &= A(4 - n^2) \cos nx + B(4 - n^2) \sin nx
\end{aligned}$$

Equating coeffients we have

$$\begin{aligned}
0 &= A(4 - n^2) \\
1/n &= B(4 - n^2)
\end{aligned}$$

Therefore $A = 0$ and $B = 1/n(4 - n^2)$ so the particular solution is:

$$\frac{\sin nx}{n(4 - n^2)}$$

Note that this solution does not work for $n = 2$.

4 The case $n = 2$

Lets plug our guess into our equation (add the two equations above the line)

$$\begin{aligned}
y &= x(A\cos 2x + B\sin 2x) \\
y' &= x(-2A\sin 2x + 2B\cos 2x) + A\cos nx + B\sin nx \\
y'' &= x(-4A\cos 2x - 4B\sin 2x) + 2(-2A\sin 2x + 2B\cos 2x) \\
4y &= x(4A\cos 2x + 4B\sin 2x) \\
\hline
\frac{1}{2}\sin 2x &= -4A\sin 2x + 4B\cos 2x
\end{aligned}$$

Equating coeffients we have

$$\begin{aligned}
1/2 &= -4A \\
0 &= 4B
\end{aligned}$$

Therefore $A = -1/8$ and $B = 0$ so the particular solution is

$$\frac{-x\cos 2x}{8}$$

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5 General Solution

We break out the $n = 1$ and $n = 2$ cases.

$$y = C_1\cos 2x + C_2\sin 2x + \frac{\sin x}{3} - \frac{x\cos 2x}{8} + \sum_{n=3}^{\infty}\frac{\sin nx}{n(4-n^2)}$$

6 Laplace Transforms

Let's solve the same problem with Laplace transforms. The hard part of Laplace is remembering the formulas. For this problem we need several transforms. To start the problem we need at least this table:

$$\begin{aligned}
\mathcal{L}\{y(x)\} &= Y(s) \\
\mathcal{L}\{y''(x)\} &= s^2Y(s) + sy(0) + y'(0) \\
\mathcal{L}\{\sin ax\} &= \frac{a}{s^2+a^2}
\end{aligned}$$

So taking the Laplace transform of $y'' + 4y = \sin nx/n$ we get

$$s^2Y(s) + sy(0) + y'(0) + 4Y(s) = \frac{1}{n}\frac{n}{s^2+n^2} = \frac{1}{s^2+n^2}$$

$$(s^2+4)Y(s) = \frac{1}{s^2+n^2} - sy(0) - y'(0)$$

$$Y(s) = \frac{1}{(s^2+n^2)(s^2+4)} - \frac{sy(0)+y'(0)}{s^2+4}$$

The last term will eventually be $C_1\cos 2x + C_2\sin 2x$ for some constants C_1 and C_2 (but perhaps not the same as in the first section). We do partial fractions of the other (first) term.

7 The case $n \neq 2$

It is time to do the partial fractions.

$$\frac{1}{(s^2+n^2)(s^2+4)} = \frac{As+B}{s^2+n^2} + \frac{Cs+D}{s^2+4}$$

Cross mutiply to obtain

$$1 = (As+B)(s^2+4) + (Cs+D)(s^2+n^2)$$
$$1 = (A+C)s^3 + (B+D)s^2 + (4A+Cn^2)s + (4B+Dn^2)$$

equating coefficients

$$\begin{aligned}
0 &= A + C \\
0 &= B + D \\
0 &= 4A + Cn^2 \\
1 &= 4B + Dn^2
\end{aligned}$$

If $n = 2$ then the third equation would be 4 times the first, but it is not. So -4 times the first added to the third, yields $0 = C(n^2-4)$: $C = 0$. The first equation now yields $A = 0$. Now -4 times the second equation added to the fourth yields $1 = D(n^2-4)$: $D = 1/(n^2-4)$. The second equation now yields $B = -1/(n^2-4)$. Therefore

$$Y(s) = \frac{-1}{n^2-4}\frac{1}{s^2+n^2} + \frac{1}{n^2-4}\frac{1}{s^2+4} - \frac{sy(0)+y'(0)}{s^2+4}$$
$$Y(s) = \frac{-1}{(n^2-4)n}\frac{n}{s^2+n^2} + \frac{K_1 s + 2K_2}{s^2+4}$$

And taking the inverse transform we get $y(x) = \sin nx/(n(4-n^2) +$ homogeneous solution part

8 The case $n = 2$

This time the partial fractions looks like

$$\frac{1}{(s^2+4)^2}$$

Hmmm ... this is not in my usual table of Laplace transforms, but it is #21 on page 265 of our text and has an inverse transform of

$$\frac{1}{16}(\sin 2x - 2x\cos 2x) = -\frac{x\cos 2x}{8} + \text{homogeneous solution part}$$

It isn't one I would remember. My usual table of Laplace transforms does have

$$\mathcal{L}\left\{xy(x)\right\} = -Y'(s)$$
$$\mathcal{L}\left\{\cos ax\right\} = \frac{s}{s^2+a^2}$$

Hence

$$\mathcal{L}\left\{x\cos ax\right\} = \frac{s^2-a^2}{(s^2+a^2)^2}$$
$$\mathcal{L}\left\{x\cos ax \pm \frac{1}{a}\sin ax\right\} = \frac{s^2-a^2}{(s^2+a^2)^2} \pm \frac{1}{s^2+a^2}$$

So algebra will cancel either the s^2 or the a^2 term on top. Ouch you have to almost know the answer, to get the answer.

So we can derive the same answer as undetermined coefficients. We get the same non-homogeneous part to the final solution and so

$$y = C_1 \cos 2x + C_2 \sin 2x + \frac{\sin x}{3} - \frac{x \cos 2x}{8} + \sum_{n=3}^{\infty} \frac{\sin nx}{n(4 - n^2)}$$