Extra Problems and Examples

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1 Separation of Variables

Find the solution $u(x, y)$ to the following equations by separating variables.

1. $u_x + u_y = 0$
2. $u_x - u_y = 0$
   answer: $u = ce^{k(x+y)}$
3. $y^2 u_x - x^2 u_y = 0$
4. $u_x + u_y = (x + y)u$
   answer: $u = c \exp\left[\frac{1}{2}(x^2 + y^2) + k(x - y)\right]$
5. $u_{xx} + u_{yy} = 0$
6. $u_{xy} - u = 0$
   answer: $u = c \exp(kx + y/k)$
7. $u_{xx} - u_{yy} = 0$
8. $xu_{xy} - 2yu = 0$
   answer: $u = x^k e^{-y^2/k}$

Solution to #4 above. Let $u = X(x)Y(y)$, plugging to the equation gives

$$X'(x)Y(y) + X(x)Y'(y) = (x + y)X(x)Y(y)$$

$$\frac{X'(x)}{X(x)} + \frac{Y'(y)}{Y(y)} = (x + y)$$

$$\frac{X'(x)}{X(x)} - x = k = y - \frac{Y'(y)}{Y(y)}$$

for some constant k. We have two ODE to solve

$$X'(x) - (x + k)X(x) = 0 \quad \text{and} \quad Y'(y) - (y - k)Y(y) = 0$$

The first has an integrating factor of $\exp(-x^2/2 - kx)$ and solution $X(x) = C \exp(x^2/2 + kx)$. The second has an integrating factor of $\exp(-y^2/2 + ky)$ and solution $Y(y) = C \exp(y^2/2 - ky)$. Multiplying the ODE solutions gives the answer above.

Solution to #7. $u = X(x)(Y(y)$

$$X''(x)Y(y) - X(x)Y''(y) = 0$$

$$\frac{X''(x)}{X(x)} = k = \frac{Y''(y)}{Y(y)}$$

$$X''(x) - kX(x) = 0 \qquad Y''(y) - kY(y) = 0$$

Supposing $k \neq 0$, we get $X(x) = C_1 e^{\omega x} + C_2 e^{-\omega x}$ and $Y(y) = C_1 e^{\omega y} + C_2 e^{-\omega y}$, where ω is the (possibly complex) number so that $\omega^2 = k$. Our answer has 4 terms

$$u = A\exp(\omega(x+y)) + B\exp(\omega(x-y)) + C\exp(\omega(y-x)) + D\exp(-\omega(x+y))$$

If $k < 0$ and changing ω so that $k = -\omega^2$ we have the alternate solution $X(x) = C_1 \cos\omega x + C_2 \sin\omega y$ and $Y(y) = C_1 \cos\omega y + C_2 \sin\omega y$ Our answer has four different terms

$$u = A\cos\omega x\cos\omega y + B\cos\omega x\sin\omega y + C\sin\omega x\cos\omega y + D\sin\omega x\sin\omega y$$

Finally if $k = 0$, $X(x) = C_1 x + C_2$ and $Y(y) = C_1 y + C_2$ giving the solution

$$u = Axy + Bx + Cy + D$$

2 Characteristic examples, Normal form table

If the PDE is $au_{xx} + bu_{xy} + cu_{yy} = 0$ and the roots of $ax^2 - bx + c$ are r and s. (Note the sign change from b in the PDE to $-b$ in the polynomial.) The constant coefficient case looks like:

Type	Hyperbolic	Parabolic	Elliptic
Roots r and s	real and $r \neq s$	real and $r = s$	complex $r = a + bi$, $s = a - bi$
Characteristics	$\Phi = y - rx$, $\Psi = y - sx$	$\Phi = \Psi = y - rx$	$\Phi = y - rx$, $\Phi = y - sx$
New variables	$\xi = y - rx$, $\eta = y - sx$	$\xi = x$, $\eta = y - rx$	$\xi = y - ax$, $\eta = bx$
Solution	$u = f(y - rx) + g(y - sx)$	$u = f(y - rx) + xg(y - rx)$	$u = f(y - rx) + g(y - sx)$
Normal form	$u_{\xi\eta} = 0$ or $u_{\xi\xi} - u_{\eta\eta} = 0$	$u_{\eta\eta} = 0$	$u_{\xi\xi} + u_{\eta\eta} = 0$

Some motivation for why this works.

Of course the most interesting question is why the sign change? It is not hard to check that $ax^2 + bx + c$ and $ax^2 - bx + c$ have the roots that are negative of each other. So if r and s are roots of $ax^2 - bx + c$ then $-r$ and $-s$ are roots of $ax^2 + bx + c$. Eventually this means $ax^2 + bx + c = a(x + r)(x + s)$. Symbolically we can write

$$a\left(\frac{\partial}{\partial x} + r\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + s\frac{\partial}{\partial y}\right)u = au_{xx} + bu_{xy} + cu_{yy} = 0$$

If you look at $u_x + ru_y = 0$, this says that the directional derivation of u in the $\langle 1, r\rangle$ direction is always zero. So u is constant along lines perpendicular to $\langle -r, 1\rangle$, that is u is constant on lines of the form $y - rx = C$ for some constant C. This change of sign reflects the change from the direction to the normal direction.

3 Characteristic examples, Normal form problems

- We do the wave equation first $c^2 u_{xx} - u_{yy} = 0$. Step 1: $A = c^2$, $B = 0$, $C = -1$ and thus $AC - B^2 = -c^2 < 0$ so the equation is hyperbolic.

  Step 2: is the find the characteristics, we need to solve

$$A\left(\frac{dy}{dx}\right)^2 - 2B\frac{dy}{dx} + C = 0$$

$$c^2\left(\frac{dy}{dx}\right)^2 - 1 = 0$$

$$\frac{dy}{dx} = \pm 1/c$$

  Which gives $y = x/c + C$ and $y = -x/c + C$ so $\Phi = x - cy$ and $\Psi = x + cy$ are the characterics.

Step 3: We solve the equation as $u = f(x - cy) + g(x + cy)$ Check that it solves the equation.

Step 4: Transforms $\xi = x - cy$ and $\eta = x + cy$ gives $u_x = u_\xi + u_\eta$, $u_y = -cu_\xi + cu_\eta$, $u_{xx} = u_{\xi\xi} + u_{\xi\eta} + u_{\eta\xi} + u_{\eta\eta}$, $u_{yy} = c^2u_{\xi\xi} - c^2u_{\xi\eta} - c^2u_{\eta\xi} + c^2u_{\eta\eta}$, So

$$c^2u_{xx} - u_{yy} = 4c^2u_{\xi\eta}$$

and the equation has the canonical form $u_{\xi\eta} = 0$

- Problem #13 in §12.4 gives the PDE $u_{xx} + 9u_{yy}$ and asks us to find the type, transform to normal form and solve. Step 1 is to classify the equation, clearly $A = 1$, $B = 0$ and $C = 9$ so that $AC - B^2 = 9 > 0$ and the equation is elliptic.

  Step 2 is to find the characterics, we need to solve

$$A\left(\frac{dy}{dx}\right)^2 - 2B\frac{dy}{dx} + C = 0$$

$$\left(\frac{dy}{dx}\right)^2 + 9 = 0$$

$$\frac{dy}{dx} = \pm 3i$$

  Which gives $y = 3ix$ and $y = -3ix$, we write these as $\Phi = y - 3ix$ and $\Psi = y + 3ix$ as characteristics.

  Step 3 from the characteristics, we can solve the equation as

$$u(x, y) = f(y - 3ix) + g(y + 3ix)$$

  Note assuming complex variables behave

$$u_{xx} = (-3i)^2f''(y - 3ix) + (3i)^2g''(y + 3ix) = -9f'' - 9g''$$

$$u_{yy} = f''(y - 3ix) + g''(y + 3ix) = f'' + g''$$

  and clearly $u_{xx} + 9u_{yy} = 0$.

  Step 4, we use the transformations $\xi = (\Phi + \Psi)/2 = y$ and $\eta = (\Phi - \Psi)/2i = 3x$ to change the PDE to the canonical form $u_{\xi\xi} + u_{\eta\eta} = 0$. Eventually $u_{\xi\xi} = u_{yy}$ and $9u_{\eta\eta} = u_{xx}$.

  The change rule was use in step 4.

$$u_x = u_\xi\xi_x + u_\eta\eta_x = 0u_\xi + 3u_\eta = 3u_\eta$$

$$u_{xx} = 3(u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x) = 9u_{\eta\eta}$$

- Problem #15 $u_{xx} + 2u_{xy} + u_{yy} = 0$ Step 1 $A = B = C = 1$, so that $AC - B^2 = 0$ and the equation is parabolic.

  Step2:

$$A\left(\frac{dy}{dx}\right)^2 - 2B\frac{dy}{dx} + C = 0$$

$$\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx} + 1 = 0$$

  factors to $(\frac{dy}{dx}) - 1)^2 = 0$ and there is the one solution $y = x + C$ so $\Phi = (y - x)$ is a characteristic

  Step 3: We need two equations, the second is x times something similar to the first so $u = f(y - x) + xg(y - x)$ (An early verion of this handout had $f(y - x) + Cx$ which is also a solution by not as general as possible. Then we had $f(y - x) + xf(y - x)$, which is inbetween, but still not as general as the current answer) Lets check it $u_x = -f'(y - x) + g(y - x) - xg'(y - x)$, $u_y = f'(y - x) + xg'(y - x)$,

$u_{xx} = f''(y-x) - g'(y-x) - g'(y-x) + xg''(y-x)$, $u_{xy} = -f''(y-x) + g'(y-x) - xg''(y-x)$ and $u_{yy} = f''(y-x) + xg''(y-x)$ so

$$u_{xx}+2u_{xy}+u_{yy} = (f''(y-x)-2g'(y-x)+xg''(y-x))+2(-f''(y-x)+g'(y-x)-xg''(y-x))+(f''(y-x)+xg''(y-x)) = 0$$

Step 4: Let $\xi = y - x$ and $\eta = x$ then $u_x = -u_\xi + u_\eta$, $u_y = u_\xi + 0u_\eta$,

$$u_{xx} = -(-u_{\xi\xi} + u_{\xi\eta}) + (-u_{\eta\xi} + u_{\eta\eta}) = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$$
$$u_{xy} = -(u_{\xi\xi} + 0u_{\xi\eta}) + (u_{\eta\xi} + 0u_{\eta\eta}) = -u_{\xi\xi} + u_{\eta\xi}$$
$$u_{yy} = u_{\xi\xi} + 0u_{\xi\eta} = u_{\xi\xi}$$
$$u_{xx} + 2u_{xy} + u_{yy} = (1-2+1)u_{\xi\xi} + 2(-1+1+0)u_{\xi\eta} + (1+0+0)u_{\eta\eta} = u_{\eta\eta}$$

And so the canonical form is $u_{\eta\eta} = 0$.

- Problem #19 Requires more steps than are in the text. It gives the PDE $xu_{xx} - yu_{xy} = 0$. Step 1 has $A = x$, $B = -y/2$ and $C = 0$, so that $AC - B^2 = -y^2/4 < 0$ (if $y \neq 0$) and the equation is hyperbolic.

  Step2:

$$A\left(\frac{dy}{dx}\right)^2 - 2B\frac{dy}{dx} + C = 0$$
$$x\left(\frac{dy}{dx}\right)^2 + y\frac{dy}{dx} = 0$$

  This factors into

$$\frac{dy}{dx}\left(x\frac{dy}{dx} + y\right) = 0$$

  The first ODE is $\frac{dy}{dx} = 0$ or $y = C$ so $\Phi = y$, the second ODE is $\frac{dy}{y} = -\frac{dx}{x}$ or $y = C/x$ or $xy = C$ so $\Psi = xy$.

  The method of the textbook does not correctly handle the next part of the problem. The method of textbook does work if A, B, C are constants. The additional work needed to solve this in this version of extra.

  Step 3: The table in the text implies $u = f(y) + g(xy)$ should be the solution. But it is not; checking we see that

$$u_x = yg'(xy); \quad u_{xx} = y^2g''(xy); \quad u_{xy} = xyg''(xy) + g'(xy)$$
$$xu_{xx} - yu_{xy} = xy^2g''(xy) - xy^2g''(xy) - yg'(xy) \neq 0$$

  Instead we need another trick.

  The trick is to let $p(x,y) = u_x$, our PDE becomes $xp_x - yp_y$ which is a first order equation and which has the general solution $p = g(xy)$ found above. (This is easy to check.) Now we just solve $u_x = g(xy)$ by integration obtaining

$$u = f(y) + \int g(xy)\,dx = f(y) + h(xy)/y$$

  Why is the $\int g(xy)\,dx = h(xy)/y$? Well it has to be something whose x-partial is a function of xy. So in must be an arbitrary function $h(xy)$ but we need to make its x-partial, $yh(xy)$, be an function of xy; clearly dividing by y does the trick. Checking this solution gives

$$u_x = yh'(xy)/y; \quad u_{xx} = yh''(xy); \quad u_{xy} = xh''(xy)$$
$$xu_{xx} - yu_{xy} = xyh''(xy) - xyh''(xy) = 0$$

  Step 4: $\xi = y$, $\eta = xy$ $u_x = 0u_\xi + yu_\eta$, $u_y = u_\xi + xu_\eta$, $u_{xx} = y(0u_{\eta\xi} + yu_{\eta\eta}) = y^2u_{\eta\eta}$, $u_{xy} = u_\eta + y(xu_{\eta\xi} + u_{\eta\eta}) = yu_{\eta\eta} + xyu_{\eta\xi} + u_\eta$, $u_{yy} = u_{\xi\xi} + xu_{\xi\eta} + x(u_{\eta\xi} + xu_{\eta\eta}) = u_{\xi\xi} + 2xu_{\eta\xi} + x^2u_{\eta\eta}$

$$xu_{xx} - yu_{xy} = xy^2u_{\eta\eta} - (y^2u_{\eta\eta} + xy^2u_{\eta\xi} + yu_\eta = xy^2u_{\eta\xi} + yu_\eta$$

  Dividing by $xy^2 = y\eta$ we get the canonical

$$u_{\eta\xi} + u_\eta/\eta = 0$$

  since the second term is lower order we are ok.