Polar/Bessel/and all that

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These are cryptic notes for Lecturing and as such are not to be completely trusted. If you see an error, please let me know. In particular this does problems 24-30 in 12.9.

1 The separation

Our PDE to solve is the wave equation $c^2(u_{xx} + u_{yy}) = u_{tt}$ in the circular region C with radius $\leq a$ with initial position and velocity $f(x, y)$ and $g(x, y)$ and $u|_{\partial C} = 0$.

We convert to polar coordinates the PDE becomes

$$c^2(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}) = u_{tt}$$

The iniatial conditions

$$u(r, \theta, 0) = f(r, \theta) \quad u_t(r, \theta, 0) = g(r, \theta)$$

and the boundary condition

$$u(a, \theta, t) = 0$$

Assume $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$ into the equation

$$c^2(R''\Theta T + \frac{1}{r}R'\Theta T + \frac{1}{r^2}R\Theta''T) = R\Theta T''$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \frac{T''}{c^2T}$$

2 The T part

Positive values for the constant are not reasonable. So Let

$$\frac{T''}{c^2T} = -\lambda^2$$

and hence when $\lambda > 0$ the function

$$T(t) = A\cos c\lambda t + B\sin c\lambda t$$

3 The Θ part

The condition on $\Theta(\theta)$ is periodicity. We must have $\Theta(0) = \Theta(2\pi)$ and $\Theta'(0) = \Theta'(2\pi)$ These requires

$$\frac{\Theta''}{\Theta} = -m^2$$

where $m = 0, 1, 2, 3, \ldots$ is an integer; and when $m > 0$

$$\Theta(\theta) = A\cos m\theta + B\sin m\theta$$

4 The R part, Bessel functions

We can rewrite the equation

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - m^2\frac{1}{r^2} = -\lambda^2$$

as

$$r^2R'' + rR' + (\lambda^2r^2 - m^2)R = 0$$

and our boundary condition is

$$R(a) = 0$$

and implied boundary condition of not being singular at $r = 0$.

Bessel's equation of order m is

$$x^2y'' + xy' + (x^2 - m^2)y = 0$$

which has a fundamental solution $y = AJ_m(x) + BY_m(x)$ where J_m is the Bessel function of the 1st kind (of order m) and Y_m is the Bessel funciont of the 2nd kind (of order m) and since Y_m is singular at $x = 0$, it will not be used here.

Our separation equation and Bessel's equation are close. Let $z = J_m(\lambda x)$ to see how to get from one to the other. We have $z' = \lambda J_m'(\lambda x)$ and $z'' = \lambda^2 J_m''(\lambda x)$ Since $y = J_m(x)$ is a solution

$$x^2J_m''(x) + xJ_m'(x) + (x^2 - m^2)J_m(x) = 0$$

Replace x by λx everywhere

$$x^2\lambda^2J_m''(\lambda x) + x\lambda J_m'(\lambda x) + (\lambda^2x^2 - m^2)J_m(\lambda x) = 0$$

Re-write in terms of z

$$x^2z'' + xz' + (\lambda^2x^2 - m^2)z = 0$$

and we see that our separation equations has

$$R(r) = AJ_m(\lambda r)$$

as its solution.

The condition $R(a) = 0$ implies that $J_m(\lambda a) = 0$ or that λa is a zero of J_m. If $k_{m,n}$ is the n th zero of J_m and $\lambda_{m,n} = k_{m,n}/a$ then (note $\lambda > 0$)

$$u_{m,n}(r,\theta,t) = J_m(\lambda_{m,n}r)\cos(m\theta)(A_{m,n}\cos(c\lambda_{m,n}t) + B_{m,n}\sin(c\lambda_{m,n}t))$$

and

$$u^*_{m,n}(r,\theta,t) = J_m(\lambda_{m,n}r)\sin(m\theta)(A^*_{m,n}\cos(c\lambda_{m,n}t) + B^*_{m,n}\sin(c\lambda_{m,n}t))$$

are solutions to the wave equation we started. The coefficients A, B, A^* and B^* are detemined from the initial data, for example

$$A_{m,n} = \frac{2}{a^2J^2_{m+1}(k_{m,n})}\int_0^a\int_0^{2\pi} rf(r,\theta)J_m(\lambda_{m.n}r)\cos(m\theta)\,d\theta\,dr$$

We find the coefficients below (with $a_{mn} = A_{m,n}$, $a^*_{mn} = A^*_{m,n}$ $b_{mn} = B_{m,n}$, $b^*_{mn} = B^*_{m,n}$) a couple of sections below.

5 Orthogonality

We show for fixed m, the functions $y_i = J_m(\lambda_{m,i}r)$ and $y_j = J_m(\lambda_{m,j}r)$ are orthogonal (with weight function r) for $i \neq j$. By this we mean

$$\int_0^a y_i(r)y_j(r)r\,dr = 0$$

This comes from the ODE that defines J_m.

$$x^2 z'' + xz' + (\lambda^2 x^2 - m^2)z = 0$$

$$x^2 z'' + xz' - m^2 z = -\lambda^2 x^2 z$$

$$z'' + \frac{1}{x}z' - \frac{1}{x^2}m^2 z = -\lambda^2 z$$

This means $y_i(r)$ and $y_j(r)$ satisfy

$$y_i'' + \frac{1}{r}y_i' - \frac{1}{r^2}m^2 y_i = -\lambda_{m,i}^2 y_i$$

$$y_j'' + \frac{1}{r}y_j' - \frac{1}{r^2}m^2 y_j = -\lambda_{m,j}^2 y_j$$

Multiply the top equation by ry_j and the bottom by ry_i, subtract and integrate from 0 to a we get

$$(\lambda_{m,j}^2 - \lambda_{m,i}^2)\int_0^a r y_i(r) y_j(r)\,dr = \int_0^a r(y_j y_i'' - y_i y_j'') + (y_j y_i' - y_i y_j')\,dr$$

Note that

$$\frac{d}{dr}(r y_i(r) y_j'(r) - r y_i'(r) y_j(r)) = r(y_i' y_j' + y_i y_j'' - y_i'' y_j - y_i' j_j') + y_i y_j' - y_i' y_j$$

so we have an anti-derivative and

$$(\lambda_{m,j}^2 - \lambda_{m,i}^2)\int_0^a r y_i(r) y_j(r)\,dr = (a y_i(a) y_j'(a) - a y_i'(a) y_j(a)) - (0 y_i(0) y_j'(0) - 0 y_i'(0) y_j(0)) = 0$$

and since $\lambda_{m,j} \neq \lambda_{m,i}$, the orthogonally condition is true.

Note this doesn't say $J_0(\lambda_{0,1}r)$ is orthogonal to $J_1(\lambda_{1,1}r)$ because they are obviously not orthogonal. (Indeed, both are positive and continuous on $0 < r < a$ and hence their product cannot have a zero integral.)

Exercise: Use the solutions $y'' = m^2 y$ to show $\cos mx$, and $\sin nx$ are orthogonal on the interval $[-\pi, \pi]$.

6 Fourier-Bessel Series

Now we need to satisfy the initial position and velocity. We need to treat the terms $J_m(\lambda_{m,n}r)\cos(m\theta)$ and $J_m(\lambda_{m,n}r)\sin(m\theta)$ like double fourier series in the section before. The integral is slightly different than a straight forward generalization would imply. There is a weighting factor of r in the integral.

The orthogonality condition drives this. There is the question of completeness. But if $f(r)$ can be written as $\sum c_n J_m(\lambda_{m,n}r)$ the orthogonality says

$$c_n \int_0^a r J_m^2(\lambda_{m,n})\,dr = \int_0^a r f(r) J_m(\lambda_{m,n}r)\,dr$$

Eventually,

$$\int_0^a r J_m^2(\lambda_{m,n})\,dr = \frac{1}{2}a^2 J_{m+1}(\alpha_{mn})$$

7 Asmar

The zero's of J_m are α_{mn} where $\alpha_{m1} < \alpha_{m2} \ldots$ and $\lambda_{mn}a = \alpha_{mn}$

$$f(r,\theta) = a_0(r) + \sum_{m=1}^{\infty}(a_m(r)\cos m\theta + b_m(r)\sin m\theta)$$

$$a_0(r) = \sum_{n=1}^{\infty} a_{0n} J_0(\lambda_{0n} r)$$

$$a_m(r) = \sum_{n=1}^{\infty} a_{mn} J_0(\lambda_{mn} r)$$

$$b_m(r) = \sum_{n=1}^{\infty} b_{mn} J_0(\lambda_{mn} r)$$

$$a_0(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r,\theta)\, d\theta$$

$$a_m(r) = \frac{1}{\pi} \int_0^{2\pi} f(r,\theta) \cos m\theta\, d\theta$$

$$b_m(r) = \frac{1}{\pi} \int_0^{2\pi} f(r,\theta) \sin m\theta\, d\theta$$

$$a_{0n} = \frac{1}{\pi a^2 J_1^2(\alpha_{0n})} \int_0^a a_0(r) J_0(\lambda_{0n} r) r\, dr$$

$$a_{mn} = \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mn})} \int_0^a a_m(r) J_m(\lambda_{mn} r) r\, dr$$

$$b_{mn} = \frac{2}{\pi a^2 J_{m+1}^2(\alpha_{mn})} \int_0^a b_m(r) J_m(\lambda_{mn} r) r\, dr$$

$$a_{0n}^* = \frac{1}{\pi c \alpha_{0n} a J_1^2(\alpha_{0n})} \int_0^a \int_0^{2\pi} g(r,\theta) J_0(\lambda_{0n} r) r\, d\theta dr$$

$$a_{mn}^* = \frac{2}{\pi c \alpha_{mn} a J_{m+1}^2(\alpha_{mn})} \int_0^a \int_0^{2\pi} g(r,\theta) J_m(\lambda_{mn} r) r\, d\theta dr$$

$$b_{mn}^* = \frac{2}{\pi c \alpha_{mn} a J_{m+1}^2(\alpha_{mn})} \int_0^a \int_0^{2\pi} g(r,\theta) J_m(\lambda_{mn} r) r\, d\theta dr$$

8 Ways to feel comfortable about $J_m(x)$

$$J_m(x) = \frac{1}{\pi} \int_0^{\pi} \cos(m\theta - x \sin\theta)\, d\theta$$

For large x

$$J_m(x) \sim \sqrt{\frac{2}{\pi x}} \cos(x - \frac{m\pi}{2} - \frac{\pi}{4})$$

and for small x

$$J_m(x) \sim \frac{1}{2^m m!} x^m$$