Computation Linear Algebra; What is it all about?

Linear Algebra is a hard class for many students to 'get the big picture'. It is not a surprise, the subject is huge. When I was a student my instructor said fully two-thirds of mathematics is about linear algebra. Most scientists and engineers end up learning a lot about linear algebra but this knowledge is collected over several courses.

However, embracing linear algebra requires accepting new ways to solving old problems. Sure the old methods will work on simple problems, but will they work on big problems? Eventually even reasonable problems became boring without the help of a computer.

Adding a computer introduces round off error and makes dramatic changes in the theory. While mathematically dividing by 3 can be reversed by multiplying by 3, numerically (on a computer) it cannot be exactly reversed when using programs like matlab.

One key notion in Linear Algebra is *dimension.* This is dimension like the line is one dimensional and space is three dimensional. We will not stop at three dimensions, we will talk about n dimensions which is often written $\mathbb{R}^n$ and even function spaces which are ∞-dimensional. The objects that have dimension are called vector spaces. (Another common but different use of dimension is when you ask for the dimension of a object, the lengths with units.)

Dimension is roughly the number of degrees of freedom. A point in $\mathbb{R}^3$ has three degrees of freedom. Any sort of direction is a combination of the three basic directions. A point in a plane can only move in two independent directions and remain in the plane. To move outside the plane, one moves in a perpendicular direction. All of these terms are made precise in linear algebra. (Linear combination, linear independence, linear basis, linear span are terms that arise.)

Geometry and length are part of vector spaces. The notation of a dot product generalizes and gives lengths (norms) and angles (between two vectors). This means one can ask best approximation questions like: What quadratic polynomial best approximates $\cos(x)$ on $[-\pi, \pi]$.

A calculus 3 problem: does $\langle 2, 3, 1\rangle$ lie in plane determined by the two vectors $\langle 1, 1, 1\rangle$ and $\langle -1, -2, 0\rangle$? This is asking if first vector is an new degree of freedom, or if it can already be expressed by to other two vectors. In Calculus 3 this three dimensional problem is solved by the triple product, but the triple product is unique to three dimensions, it does not generalize.

A second key notion is a *linear transformation.* This is an operator L with the two properties

1. $L(x + y) = L(x) + L(y)$
2. $L(\alpha x) = \alpha L(x)$

Examples of linear transformations include

1. Matrix multiplication
2. Rotations, Rescaling, Skewing, but not translation
3. Differentiation, and Integration

4. Many Ordinary and Partial Differential equations are linear.

The range and domain of a linear transformation is a vector space.

Linear transformations and degrees of freedom often mix in a very nice manner. There is often a natural basis which reduces the linear transformation to a diagonal map. This is like a matrix with only entries on the diagonal. The natural basis is made up of eigenvectors. Essentially we have UN-coupled a coupled equation.

Solving the differential equation $y'' + y = 0$ is the same as finding the solutions to the transformation $(D^2 - I)f = 0$ (where $Df = f'$). The language of linear algebra is used in differential equations.

Mathematically we have complete answers to equations like

1. Does $Lx = y$ have a solution?

2. Does $Lx = y$ have a unique solution?

3. How do we find all solution to $Lx = y$?

4. If we move y little bit, does that move the solution to $Lx = y$ a little bit?