Practice Mini-Test 1 and answers – Calculus 3 – Spring 04

1. Q3 F03 Plot the points $P(3,5,-1)$ and $Q(-3,3,5)$ on a 3D graph (whose axeses are in the usual positions). Draw the vector $\overrightarrow{PQ}$ on the graph and write $\overrightarrow{PQ}$ in the $\langle ?,?,?\rangle$ notation.

   See `http://www.math.fsu.edu/~bellenot/class/f03/quiz/q3a28.pdf`.

2. T1#4 S02 Find the center and radius of the sphere S given by the equation $x^2+y^2+z^2+2x+8y-4z=28$. The graph of S intersects the xz-plane in a circle, what is its equation, its center and its radius. [compare T1#1 F03]

   $(x^2+2x+1)+(y^2+8y+16)+(z^2-4z+4)=28+1+16+4$
   $(x+1)^2+(y+4)^2+(z-2)^2=7^2$

   The sphere has center $(-1,-4,2)$ and radius 7. The xz-plane is $y=0$ so $(x+1)^2+16+(z-2)^2=49$ or $(x+1)^2+(z-2)^2=(\sqrt{33})^2$ so center at $(-1,0,2)$ and radius $\sqrt{33}$.

3. T1#1 S03 Find the equation of the plane parallel to the plane $3x-4y-6z=21$ and passing through the point $(-3,1,2)$ and find the distance between the two parallel planes. [compare T1#1 F02]

   Equation of the plane is $3x-4y-6z=3(-3)-4(1)-6(2)=-9-4-12=-25$.

   Distance is $|3(-3)-4(1)-6(2)-21|/\sqrt{3^2+4^2+6^2}=46/\sqrt{61}$

4. T1#2 F03 Find the equation of the plane through the points $(2,1,-2)$, $(3,-1,2)$ and $(4,0,1)$. [compare T1#2 S03, T1#2 F02]

   Let $A(2,1,-2), B(3,-1,2)$ and $C(4,0,1)$. Then $\overrightarrow{AB}=\langle 1,-2,4\rangle$ and $\overrightarrow{AC}=\langle 2,-1,3\rangle$ so the cross product $\overrightarrow{AB}\times\overrightarrow{AC}$ is $\langle -2,5,3\rangle$. Checking:
   $\langle -2,5,3\rangle\cdot\langle 1,-2,4\rangle=-2-10+12=0\checkmark$ $\quad\langle -2,5,3\rangle\cdot\langle 2,-1,3\rangle=-4-5+9=0\checkmark$
   So the equation is $-2x+5y+3z=-2(2)+5(1)+3(-2)=-5$

5. T1#3 F03 Let $P(3,-2,2)$ and $\vec{v}=\langle 3,-1,5\rangle$, find:

   (a) The equation of the line through P in the direction of $\vec{v}$
   The equation in parametric form: $x=3+3t, y=-2-t, z=2+5t$

   (b) The coordinates of the point where the line in (a) intersects the xz-plane.
   The xz-plane is where $y=0$, and $0=-2-t$ implies $t=-2$ which yields the point $(-3,0,-8)$.

   (c) The equation of the plane perpendicular to $\vec{v}$ through P.
   The equation is $3x-y+5z=3(3)-(-2)+5(2)=21$

   (d) The coordinates of the point where the y-axis intersects the plane in (c).
   The y-axis has $x=0$ and $z=0$ hence $3(0)-y+5(0)=21$ or the point is $(0,-21,0)$.

6. T1#6 F03 A treasure map reads start at the big X, walk 40 paces north, 20 paces northwest and dig a hole 10 paces deep. Write the vector $\vec{v}$ that goes from the big X to the bottom of the hole and find the exact simplified value of the length squared $\|\vec{v}\|^2$. (The x-axis points East, the y-axis points North, and the z-axis points up.) [compare T1#3 S02, T1#6 F02]

   A vector addition problem. The segments are given by the vectors $\langle 0,40,0\rangle$, $\langle -20/\sqrt{2},20/\sqrt{2},0\rangle$, and $\langle 0,0,-10\rangle$, so $\vec{v}=\langle -20/\sqrt{2},40+20/\sqrt{2},-10\rangle$ and $\|\vec{v}\|^2=400/2+1600+1600/\sqrt{2}+400/2+100=2100+800\sqrt{2}$.

7. T1#8 F03 Using vector operations write $\vec{a}=\langle 2,-1,5\rangle$ as the sum of two vectors, one parallel (say $\vec{v}$), and one perpendicular (say $\vec{w}$) to $\vec{b}=\langle -4,4,2\rangle$. [compare T1#8 S03, T1#8 F02]

   We need a unit vector $\vec{u}$ in the direction of $\vec{b}$ which has length 6 so $\vec{u}=\langle -2/3,2/3,1/3\rangle$. Now $\vec{u}\cdot\vec{a}=-4/3-2/3+5/3=-1/3$ so $\vec{a}_{||}=\vec{v}=\langle 2/9,-2/9,-1/9\rangle$ and $\vec{a}_{\perp}=\vec{w}=\langle 16/9,-7/9,46/9\rangle$. Check $\vec{b}\cdot\vec{w}=-64/9-28/9+92/9=0\checkmark$.

8. T1#6 S03 Determine if the lines L_1 and L_2 are parallel, skew or intersecting. If they intersect, find the point of intersection.
$L_1: \quad x = 2 + t, y = 2 - t, z = 5 + 3t$
$L_2: \quad x = 1 - s, y = 1 + 2s, z = -6 + s$ [compare T1#4 F02]

   We get two equations from $x = x$ and $y = y$, namely $2 + t = 1 - s$ and $2 - t = 1 + 2s$ and solving we get $s = 2$ and $t = -3$. When $t = -3$ the point on L_1 is $(-1, 5, -4)$. When $s = 2$ the point on L_2 is $(-1, 5, -4)$. Since the points agree the lines are intersecting and the point of intersection is $(-1, 5, -4)$.

9. T1#7 S03 Find the parametric equation of the line through the points $P(3, 2, 8)$ and $Q(4, 4, -4)$ and find the two points where it it intersects the elliptical paraboloid $z = x^2 + y^2$. [compare T1#10 S02]

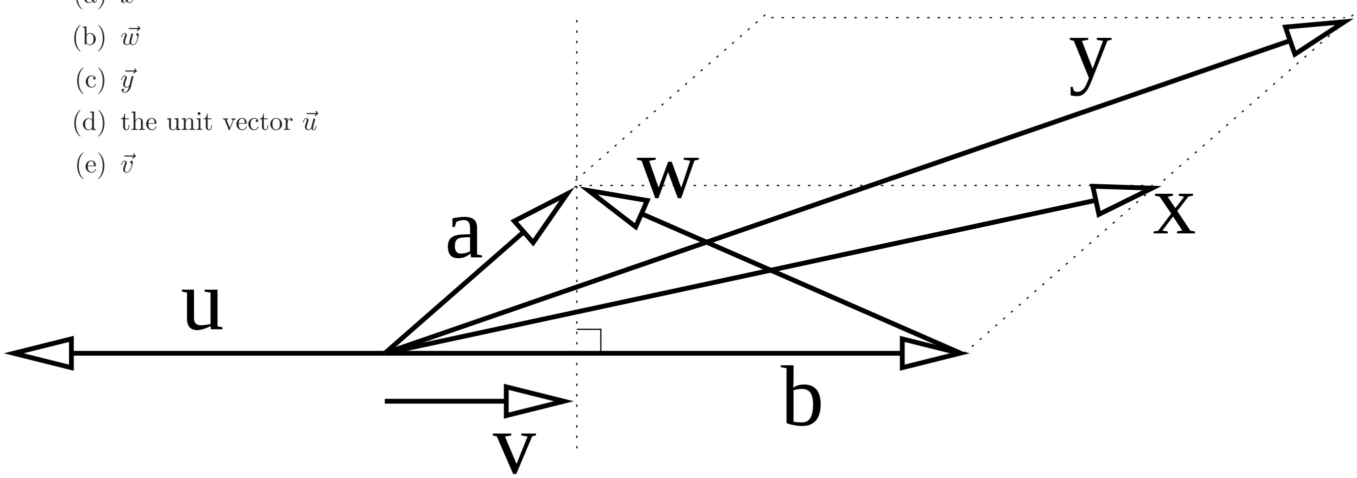
   Velocity vector $\overrightarrow{PQ} = \langle 1, 2, -12 \rangle$ and parametric equations $x = 3 + t, y = 2 + 2t, z = 8 - 12t$. Solving for t in $8 - 12t = (3 + t)^2 + (2 + 2t)^2$ or $5t^2 + 26t + 5 = 0$ which factors to $(5t + 1)(t + 5) = 0$. When $t = -5$ the point is $(-2, -8, 68)$ and when $t = -1/5$ the point is $(-14/5, 8/5, 52/5)$. Checking $(-2)^2 + (-8)^2 = 4 + 64 = 68$ ✓ and $(-14/5)^2 + (8/5)^2 = 196/25 + 64/25 = 260/25 = 52/5$ ✓.

10. T1#9 F03 Find parametric equations of the line of intesection of the two planes $x + 2y + 2z = 3$ and $3x + 2y - 2z = 9$.

    Find two points, first let $x = 0$ and solve both $2y + 2z = 3$ and $2y - 2z = 9$. The first point is (adding gives $4y = 12$ so $y = 3$ and $z = -3/2$) $(0, 3, -3/2)$. Now let $y = 0$ and solve both $x + 2z = 3$ and $3x - 2z = 9$. Again (adding gives $4x = 12$ so $x = 3$ and $z = 0$) we get a second point $(3, 0, 0)$. This gives a velocity vector $\vec{v} = \langle 3, -3, 3/2 \rangle$ and hence parametric equations $x = 3t, y = 3 - 3t, z = -3/2 + 3t/2$.

11. T1#3 S02 For the given vector, write it as an expression in terms of the vectors $\vec{a}$ and $\vec{b}$ suggested by the picture below.

    (a) $\vec{x}$
    (b) $\vec{w}$
    (c) $\vec{y}$
    (d) the unit vector $\vec{u}$
    (e) $\vec{v}$



Answers $\vec{x} = \vec{a} + \vec{b}$, $\vec{w} = \vec{a} - \vec{b}$, $\vec{y} = 2\vec{a} + \vec{b}$, $\vec{u} = -\vec{b}/\|\vec{b}\|$ and $\vec{v} = \left(\vec{a} \cdot (\vec{b}/\|\vec{b}\|)\right)(\vec{b}/\|\vec{b}\|) = (\vec{a} \cdot \vec{b}/\|\vec{b}\|^2)\vec{b}$ the vector projection of $\vec{a}$ in the direction of $\vec{b}$.