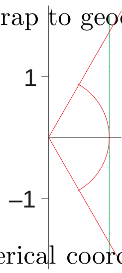
Geodesics — Project Part 5

We consider two more surfaces. The first is the cone, which is like like the cylinder in sense one can wrap a piece of paper around it. The second is the hyperbolic paraboloid, our favorite saddle surface, which is not wrapable, but does have lots of straight lines for geodesics. Some other geodesics of our saddle will be found using a maple program (see project web site).

Basically the geodesics for the cone can be obtained by drawing a straight on a piece of paper and wrapping in around the cone (see the picture below) The orgin on the paper is the vertex of the cone and the paper wraps several times around the cone. Here the two red angles will map to the same line on the back side of the cone. The green line will wrap to geodesic on the cone. (More pictures on the project web site.)



For the cone we will use polar and spherical coordinates. The cone is given by the spherical equation $\phi = \alpha$. In the flatten out cone, the line is given by $x = 1$ $y = t$. It is not hard to convert this to a parametric polar form with $r = \sqrt{1+t^2}$ and $\theta = \arctan(t)$. When this gets wrapped around the cone, the r of polar coordinates is mapped to ρ of spherical coordinates, but the expression for θ in spherical in terms of θ for the paper is not equality but depends on α-angle of the cone, and is $\theta_{\text{spherical}} = \theta_{\text{polar}}/\sin(\alpha)$.

The hyperbolic paraboloid $z = a^2x^2 - b^2y^2$, cannot be flatten out like the cone and the cylinder, but it does have an large number of straight lines. (These straight lines will be geodesics.) You can actually build a hyperbolic paraboloid out of string art.

Part A. (i) Suppose $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ is any curve in space that misses the origin and suppose when $t = t_0$ the distance from the orgin to $\vec{r}$ is a global minimum. Show $\vec{r}(t_0)$ is perpendicular to the velocity $\vec{v}(t_0)$. (ii) Convert the spherical parametric equations

$$\rho = \sqrt{1+t^2}, \phi = \alpha, \theta = \arctan(t)/\sin(\alpha)$$

into rectangular coordinates. (iii) Using the special value $\alpha = \arcsin(1/2)$ show the rectangular parametric equation has constant speed.

Part B. (i) Let $a > 0$ and $b > 0$, show that for the hyperbolic paraboloid $z = f(x, y) = a^2x^2 - b^2y^2$ and any point (x_0, y_0) that both parametric equations $\vec{r}(t) = \langle x_0 + bt, y_0 + at, f(x_0 + bt, y_0 + at)\rangle$ and $\vec{r}(t) = \langle x_0 + bt, y_0 - at, f(x_0 + bt, y_0 - at)\rangle$ are parametric equations of lines (which will be geodesics). (ii) Use the maple code (see web site) to picture two nearby geodesics both in the $\vec{i}$ direction starting at $(0, 1, -1)$ and $(0, -1, -1)$. The maple code on the web site does two nearby geodesics starting at $(1, 0, 1)$ going in the $\langle -1/3, 1, ?\rangle$ direction and at $(-1, 0, 1)$ going in the $\langle 1/3, 1, ?\rangle$ direction.

Hints: A. (i) Compare the equation for t_0 a critical point with $\vec{r}(t_0) \cdot \vec{v}(t_0)$. B and the rest of A are straight forward.

Eventually all parts of the project will collected into a typed report, but this assignment does not have to be typed.

Note that **quality of presentation is extremely important**. It is not enough merely to produce an answer: the method by which you obtain it must be sound, and you must clearly demonstrate that you understand it. Therefore, there will be penalties (commensurate with degree of infraction) for bad presentation which includes bad grammar, illegibility, incompleteness, incoherence and untidiness especially on the written assignment. Even the format of is important, use one side of the paper only, with empty margins along the left side and top of each page and with multi-page answers stapled together (do NOT use dog ears). (There is a stapler in 208 Love you can use.) Unstapled assignments will not be accepted. Late assignments will not be accepted without prior approval.

Finally group projects require each member fill out a separate sheet which evaluates the relative contributions of the other members of the group. This form is due at the same time as the project.