Proofs about graphs which are almost trees.

The following statements are equivalent (for a simple graph G):

a. G is a tree (For vertices x and y of G there is a unique xy-path in G).
b. G is connected and acyclic. (Acyclic means it has no cycles.)
c. G is connected and $|E| = |V| - 1$.
d. G is acyclic and $|E| = |V| - 1$.
e. G is connected and each edge e is a bridge. (e is a bridge or cut edge means $G - e$ is disconnected.) [This says a tree is a minimal connected graph.]
f. G is acyclic and if $x, y \in V(G)$ and $e = xy \notin E(G)$ then $G + e$ has a cycle. [This says a tree is a maximal acyclic graph.]

1. Show the following statements are equivalent (G a simple graph):
a. G is connected and has exactly one cycle.
b. G is connected and $|E| = |V|$.
c. G has an edge e so that $G - e$ is connected and acyclic.
d. There is a tree T and an edge $e \notin E(T)$ so that $G = T + e$.

The a, b, c and d above are also equvalent to e below (but you are not asked to prove this.)
e. G is connected, but not a tree, and for vertices x and y of G there are at most two simple xy-paths in G.

2. Show the following statements are equivalent (G a simple graph):
a. There is an edge $e \notin E(G)$ so that $G + e$ is a tree.
b. G is acylic and $|E| = |V| - 2$.
c. G is acylic, disconnected and there is an edge so that $G + e$ is connected.
d. G is the disjoint union of two trees. [G has two components both of which are trees.] And each of the items above implies 2e, but not conversely.
e. If G is disconnected and for x, y vertices of G in different components, then for $e = xy$, $G + e$ is connected.

The following are from old tests.

4. Prove G is a forest $\iff$ every edge of G is a cut edge.

5. Prove by induction (on the number of vertices), if T is a tree, then $|V(T)| = |E(T)| + 1$.

6. Prove if G is acyclic and $|E(G)| = |V(G)| - 2$, then there is an edge $e \in E(\bar{G})$ so that $G + e$ is a tree.

7. Using the formula $\sum_{v \in V(G)} \deg v = 2|E(G)|$, or induction, or any other method, prove that a tree with a vertex of degree 3 has at least 3 vertices of degree 1. [*Hint:* the tree must have at least 4 vertices.]

8. Prove if G is connected and $|E(G)| = |V(G)|$, then there is an edge $e \in E(G)$ so that $G - e$ is a tree.