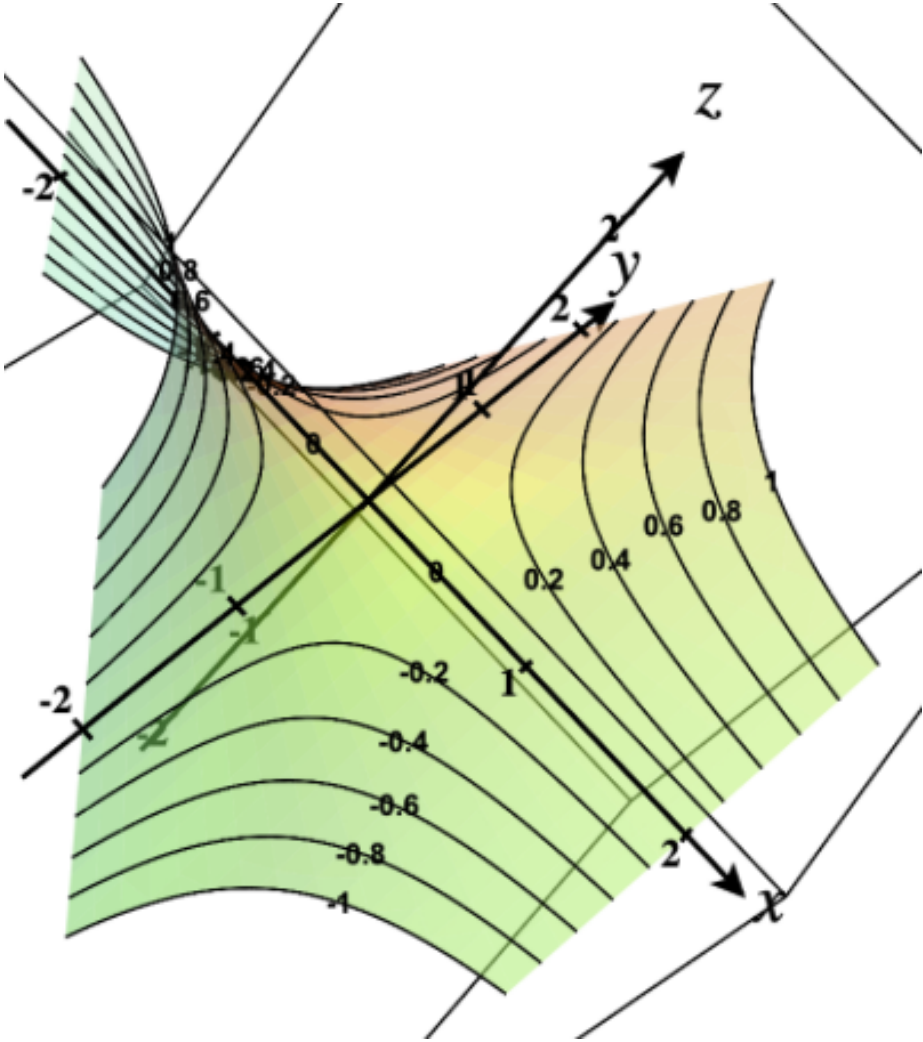


Back in the Saddle Again Halloween Draft Version

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Introduction

This project is based on some applications of saddle points. In three dimensions one can draw (2-D) *saddle surfaces* using equations like $z = x^2 - y^2$ or $z = 2xy$ (basically a rotation of $\pi/4$). [These are the Real part and the Imaginary part of $(x+iy)^2$.] There are also saddles in higher dimensions, for example the equation $w = x^2 + y^2 - z^2$, has a saddle at $(0, 0, 0)$.

Our first collection of applications are about interpolation. Often *interpolation* is a way of approximating a value, given known values around it. This is used in data science, numerical analysis and as we will see later, even game theory.

1 Linear Interpolation

We are given the values of a function $f(x)$ at $x = 0$ and $x = 1$, say $f(0) = y_0$ and $f(1) = y_1$, and want to interpolate the value of f for any value inbetween 0 and 1. A straight forward equation is

$$f(x) = (1 - x)y_0 + xy_1 \quad (*)$$

which is clearly linear, $f(0) = y_0$ and $f(1) = y_1$. Thus it is the line throught $(0, y_0)$ and $(1, y_1)$. Note that this equation is defined for values outside the interval $[0, 1]$. The function given by $(*)$ is sometimes called the *interpolate*. If $x \notin [0, 1]$, useing $f(x) = (1 - x)y_0 + xy_1$ is called *extrapolation*, as we are using the data to compute a value outside the data. Extrapolation is likely to introduce larger errors than interpolation.

If we are given (x_0, y_0) and (x_1, y_1) , the equation can be rescaled as

$$f((1 - s)x_0 + sx_1) = (1 - s)y_0 + sy_1$$

Thus $f(x)$ can be computed by using

$$s = (x_1 - x)/(x_1 - x_0)$$

This is closely related to parametric equations of lines (in both \mathbb{R}^2 and \mathbb{R}^3 and to vector equations of lines.

This $(1-x)y_0 + xy_1$ is a convex combination of y_0 and y_1 as long as $0 \leq x \leq 1$. The convex hull of $\{y_0, y_1\}$ is the set $\{(1-x)y_0 + xy_1 : 0 \leq x \leq 1\}$ and it is the line segment between y_0 to y_1 .

2 Obsolete Technologies

In olden days, people needed to use log tables to avoid messy multiplications. A Log table often ran over https://commons.wikimedia.org/wiki/File:APN2002_Table_1,_1000-1500.agr.tiff 100 pages like this one. To gain an extra digit of accuracy, proportional parts were provided. In the image we see

$$\log 1.002 = .00087, \log 1.003 = .00130, \text{ with } 43 \text{ between them}$$

and it has a prop parts table whose entry for 43 looks like

1	4
2	9
3	13
4	17
5	22
6	26
7	30
8	34
9	39

this allows computing the logs between 1.002 and 1.003 by adding the proportional part. For eampls

$$\log 1.0024 = .00087 + 17 = 0.00104$$

$$\log 1.0027 = .00087 + 30 = 0.00117$$

Proportional part tables are just linear interpolation. For example

$$f(0.4) = (1 - 0.4)0 + 0.4(43) = 17.2 \approx 17$$

$$f(0.7) = (1 - 0.7)0 + 0.7(43) = 30.1 \approx 30$$

So now one can compute $(1.0024) * (1.0027)$ by adding the two numbers to get

$$\log \text{ product} = 0.00104 + 0.00117 = 0.00221$$

This is between

$$\log 1.005 = 0.00217$$

$$\log 1.006 = 0.00260$$

also with a jump of 43. Since $0.00221 - 0.00217$ as jump of 4 we can read the proportional parts backward to get

$$\log 1.0051 = 0.00221$$

and the product is 1.0051. Of course today, any calculator says

$$(1.0024)(1.0027) = 1.0051065,$$

but would it tell you what is the correct number of digits?

Vernier calipers also use a form on linear interpolation to gain an extra digit of accuracy. Not completely obsolete, but laser measuring devices have stolen much of the market. https://en.wikipedia.org/wiki/Vernier_scaleSee Wikipedia

Exercises

1. Repeat using the log table for the calculation $(1.0053)*(1.1003) = 1.10061316$. Make sure your answer has the correct number of digits.
2. Use (*) to Compute the interpolate $f(x)$ if $y_0 = 0, y_1 = 1$.

3 Bilinear Interpolation

This time we are given values at the four corners of the unit square. The corners are $(0,0), (1,0), (1,1)$ and $(0,1)$ and the values are z_{00}, z_{01}, z_{11} and z_{10} . Again we have a straight forward equation

$$f(x, y) = (1-x)(1-y)z_{00} + x(1-y)z_{10} + (1-x)yz_{01} + xyz_{11} \quad (**)$$

which is not linear, but has the correct values at the four corners. The equation for f is a quadratic equation since it has terms of degree two, all of which are xy .

A general two variable quadratic equation has the form

$$f(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F,$$

but bilinear interpolation has $A = C = 0$. And

$$f(x, y) = Bxy + Dx + Ey + F,$$

always gives a hyperbolic paraboloid, a saddle surface, or in the degenerate case, when $B = 0$, a plane.

An interesting geometry is exposed when we rewrite (**)

$$f(x, y) = (1-x)[(1-y)z_{00} + yz_{01}] + x[(1-y)z_{10} + yz_{11}]$$

which says for a fixed y_s , the function of x

$$f(x, y_s) = (1-x)y_0 + xy_1$$

where

$$y_0 = (1-y_s)z_{00} + y_s z_{01}$$

$$y_1 = (1-y_s)z_{10} + y_s z_{11}$$

is linear interpolation, a line segment. In particular equation (**) is a line if we fix one of the variables.

4 Now for something concrete

Lets make a ideal concrete sidewalk covering the unit square. Ideally (for drainage) there would be a slight downward tend as we move away from the x and y axis so the point z_{00} is higher then all the others, and z_{11} lower. After being poured, a straight edge (the floater), is used to level the concrete off, but make sure in fits exactly between (both pairs) of opposite sides. If done correct the top of the concrete is given by the bilinear interpolation formula. The straight edge is making sure surface is the convex combination of the opposite sides.

While we think of concrete slabs as examples of planes, this only happens if the four corners are perfectly co-planar. Nature is not so perfect. The exercises below should convince you that every slab is really a hyperbolic paraboloid!

Exercises

3. Stewart 14.4 Problem 23 used linear approximation (the tangent plane) for (Heat Index) h as a function of Temperatue, t , and Humidity, w , at $(t, w) = (94^\circ, 80\%)$, given by

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial w} dw$$

to find the heat index at $(95, 78)$, so $dt = \Delta t = 1, dw = \Delta w = -2$, the approximation was

$$127 + \frac{8}{2}(1) + \frac{5}{5}(-2) = 129,$$

where the $8/2$ and $5/5$ were numerical approximations to the partials f_t and f_w . Instead you are to use bilinear inteplation to approximate this number $f(95^\circ, 78\%)$. Note this is superior for several reasons, it is not limited to being near the tangent point, it uses more data, and numerical approximations to derivatives are less stable then interpolation.

4. Show that the equation for bilinear interpolation (***) can always be rewritten like

$$Axy + Bx + Cy + D \quad (\#\#)$$

by computing A, B, C , and D from $z_{00}, z_{01}, z_{11}, z_{10}$ in (***)

5. Use (***) to compute $f(x, y)$ for $z_{00} = 0, z_{01} = z_{10} = 1, z_{11} = 2$.
6. Show if $A \neq 0$ that the equation (***) can always be rewritten like

$$a(x - b)(y - c) + d \quad (\$\$)$$

by computing a, b, c , and d from A, B, C, D in (***) . What happens when $A = 0$?

7. Explain how to rescale and translate the equation $z = xy$ to be the equation (§§). (Hint: first multiply by a to rescale and then translate to the new position.)
8. To see that $z = xy$ is a hyperbolic paraboloid, rotate the coordinates of $z = u^2 - v^2$, by using

$$\begin{aligned}u &= x \cos \theta + y \sin \theta \\v &= -x \sin \theta + y \cos \theta\end{aligned}$$

and $\theta = \pi/4$.

9. Rewrite $z = xy$ and $z = x^2 - y^2$ in cylindrical coordinates in terms of trigonometric functions of 2θ (and not of one θ , so no $\sin \theta$).
10. Show $f(x, y) = Axy + Bx + Cy + D$ is "completely" made up of lines. For any $x_0, g(y) = f(x_0, y)$ and for any $y_0, h(x) = f(x, y_0)$ are both both lines. Do a 3D Plot with some example lines to illustrate.
11. (Critical Points are in §14.7.) Suppose $A \neq 0$. Show $f(x, y) = Axy + Bx + Cy + D$ has exactly one critical point, call it (x_c, y_c) and show that this critical point is classified as a saddle.
12. Suppose $A \neq 0$. Using the critical point you just found, show $f(x, y)$ can be rewritten

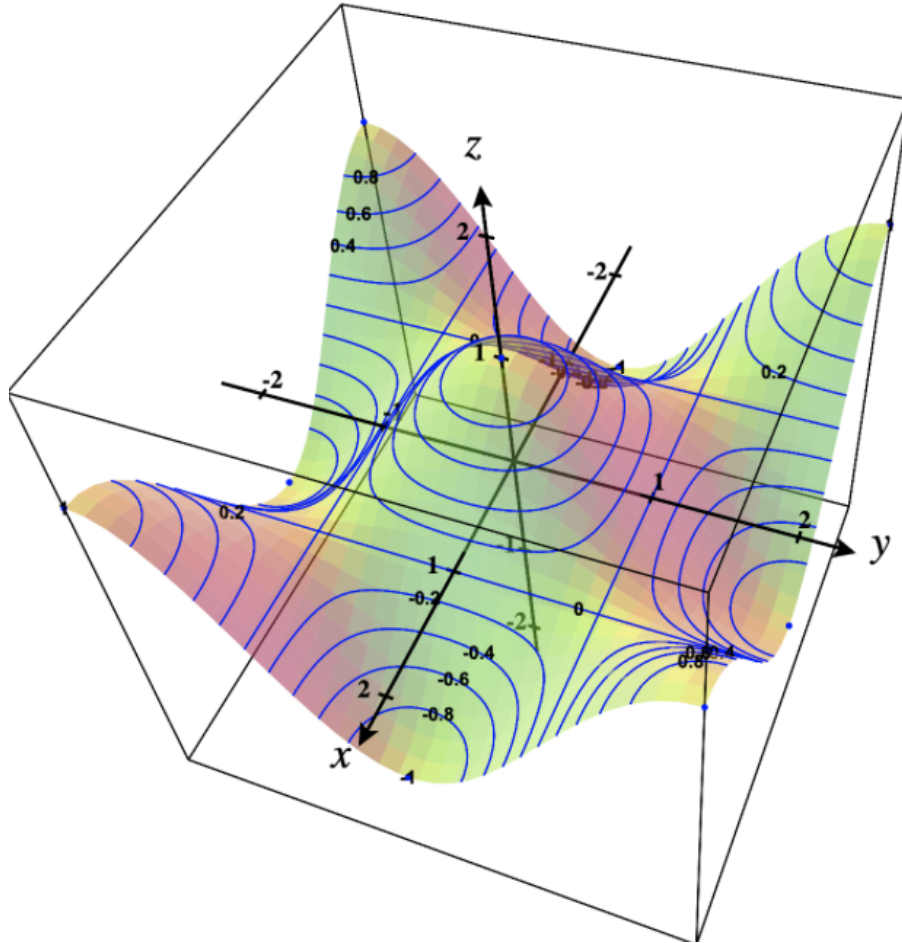
$$f(x, y) = M(x - x_c)(y - y_c) + V$$
 When this happens we will say (x_c, y_c) is *the gap point* of the hyperbolic paraboloid.
13. Plot $f(x, y) = xy$, add level curves, which are equilateral hyperbolas. A free web based CalcPlot3D has a link off our class web page. Show this is the interpolating function for $w_{00} = w_{01} = w_{10} = 0, w_{11} = 1$.
14. Use (*) to find $f(x, y)$ for $w_{00} = 1, w_{01} = w_{10} = w_{11} = 0$.

5 Data Science

Often one wants to visualize the function that likely gave you the data. One common technic is to used the data to get the inteplating function and use it to visualize the data.

For example, we can find approximations of our data's level curves by finding the same level curve of the interpolate. If our data is temperature, we are finding isotherms. If our data is pressure, we are finding isobars. If our data is elevation, it could be isoheights, but they are called contours (and wikipedia suggests isohypse).

Consider $g(x, y) = \cos(x\pi/2)\cos(y\pi/2)$ which is graphed below via CalcPlot3D. Note in the range $-1 \leq x, y \leq 1$, the graph is a nice hill, that is zero on points where either x or y is an integer.



Exercises

15. For $g(x, y) = \cos(x\pi/2)\cos(y\pi/2)$. Let $w_{00} = g(0, 0)$, $w_{01} = g(0, 1)$, $w_{10} = g(1, 0)$ and $w_{11} = g(1, 1)$. Let $f(x, y)$ be our bilinear interpolate see above. Compare the level surface for $g = 0.5$ with the level surface for $f = 0.5$. What are their equations?
16. Give a 3D plot of both f and g in the range $0 \leq x, y \leq 1$ showing the contour line for $z = 0.5$. CalcPlot3D can do such a drawing. Is the contour of f a good approximation for the contour of g when viewed from above?
17. Repeat for $g(x, y) = 1 - x^2 - y^2 = 1 - r^2$, that is find the interpolate and

plot the level curves for 0.5 for $0 \leq x, y \leq 1$

18. Many different functions can have the same interpolate, show that this is the case for these functions

$$\begin{aligned} g(x, y) &= \cos(x\pi/2)\cos(y\pi/2) \\ h(x, y) &= (1 - x^2)(1 - y^2) \\ k(x, y) &= (1 - |x|)(1 - |y|) \\ m(x, y) &= \begin{cases} 1 - y & \text{if } y \geq x \\ 1 - x & \text{if } y \leq x \end{cases} \end{aligned}$$

6 Game Theory

Zero-sum matrix games have two players: Rose, that picks the row i and Colin who picks the column j . After the selection, Colin pays a_{ij} to Rose, or if $a_{ij} < 0$, Rose pays $-a_{ij}$ to Colin. It is called zero-sum not because no money is exchanged, but because money is conserved. We will be concerned with 2×2 matrix games, with matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

When attempting to solve A , the first step is to look for saddle points. (Unfortunately, math reuses words like saddle point for different properties, so we will call these *game theory saddles*) The best way to find a game theory saddle is to put a checkmark on the largest number in each column and the smallest number in each row.

Often the best strategy is a *mixed strategy* where an element of randomness comes in. Rose plays row 1 with probability $1 - s$ and row 2 with probability s , for some $s, 0 \leq s \leq 1$. A mixed strategy with $s = 0$ (row 1) or $s = 1$ (row 2) is called a *pure strategy*. Let

$$p = \begin{bmatrix} 1 - s \\ s \end{bmatrix} \text{ and } q = \begin{bmatrix} 1 - t \\ t \end{bmatrix}$$

The result of Rose playing p and Colin playing q is

$$p^T A q = (1 - s)(1 - t)a_{11} + s(1 - t)a_{21} + (1 - s)ta_{12} + sta_{22} \quad (@@)$$

Compare with equation (**). Note the mixed strategy is sometimes the gap point. Let agree to use $\text{game}(x, y)$ for the interpolate from (**) where $w_{00} = a_{1,1}, w_{01} = a_{1,2}, w_{10} = a_{2,1}, w_{11} = a_{2,2}$ and either $\text{game}(x, y)$ has a gap point (x^*, y^*) or it is a plane.

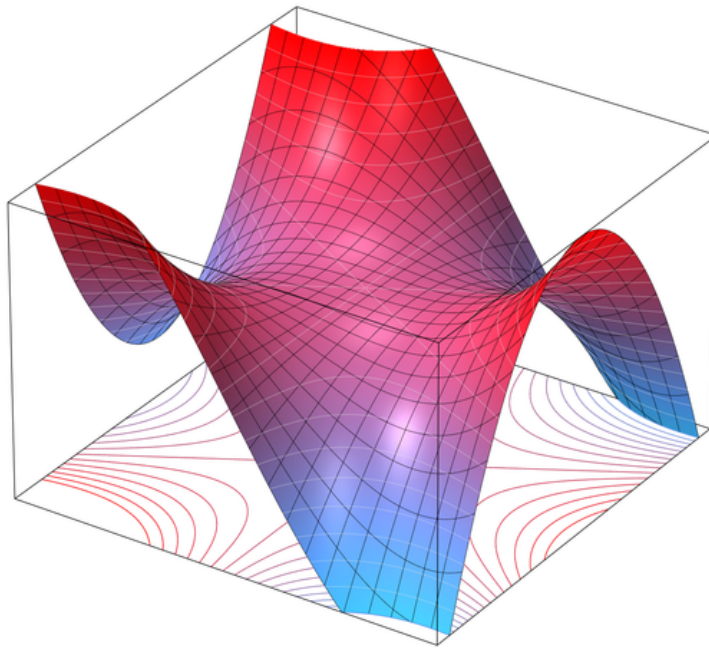
7 Exercises

These are not quite right???

19. Compute A if $\text{game}(x, y) = (x + 1/5)(y + 3/5)$, find the game saddle point.
20. Compute A if $\text{game}(x, y) = (x + 1/5)(y - 3/5)$, find the game saddle point.
21. Compute A if $\text{game}(x, y) = (x - 1/5)(y - 3/5)$, find the game value with mixed strategies

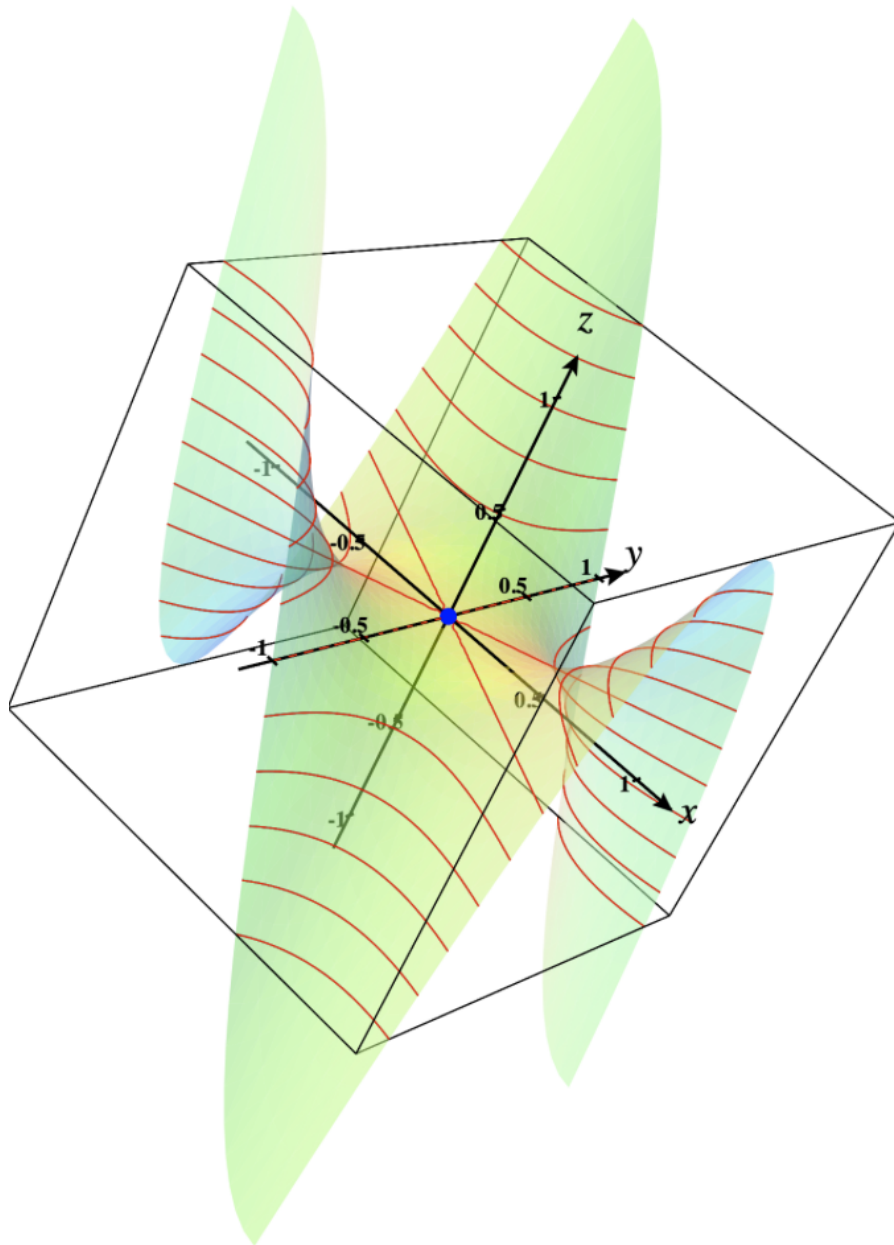
$$p = \begin{bmatrix} 4/5 \\ 1/5 \end{bmatrix} \text{ and } q = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$$

22. Compute A if $\text{game}(x, y) = (x - 1/5)(y + 3/5)$, find the game saddle point.



8 Saddles for other animals

The Monkey saddle is the surface $z = x^3 - 3xy^2$ which is pictured below. There are three down valleys, allowing for the two legs and the tail of the monkey to hang down. Note z is the real part of $(x + iy)^3$. The plot above is from Wikipedia, the second plot below is using CalcPlot3D, with $-1 \leq x, y \leq 1$ and contours without labels turned on.



Exercises

23. Convert $f(x, y) = x(x^2 - 3y^2)$ to polar coordinates. Hint compute $\cos(3\theta) = \cos(2\theta + \theta)$ in terms of $\sin \theta$ and $\cos \theta$.

24. Create a function $f(x, y)$ and give a 3D plot of a star fish saddle for a star with 5 arms.
25. Create a function $f(x, y)$ and give a 3D plot a octopus saddle for a star with 8 limbs.

9 Trilinear Interpolation

Now we are given values at the eight corners of the unit cube. The corners are $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$ on the bottom and $(0, 0, 1)$, $(1, 0, 1)$, $(1, 1, 1)$, $(0, 1, 1)$ on the top. The values at these corners are w_{000} , w_{100} , w_{110} , w_{010} , w_{001} , w_{101} , w_{111} , w_{011} . The equation is now

$$\begin{aligned}
 f(x, y, z) = & (1-x)(1-y)(1-z)w_{000} + x(1-y)(1-z)w_{100} \\
 & + xy(1-z)w_{110} + (1-x)y(1-z)w_{010} + (1-x)(1-y)zw_{001} \\
 & + x(1-y)zw_{101} + xyzw_{111} + (1-x)yzw_{011} \quad (***)
 \end{aligned}$$

which is not quadratic, but has the correct values at the eight corners. The equation for f is a cubic equation since it has terms of degree three, all of which are xyz .

A general three variable cubic equation has the form

$$\begin{aligned}
 f(x, y, z) = & Ax^3 + By^3 + Cz^3 \\
 & + Dx^2y + Ex^2z + Fy^2x + Gy^2z + Hz^2x + Iz^2y \\
 & + Jxyz + \text{terms of lower degree}
 \end{aligned}$$

but trilinear interpolation has only one degree 3 term, the xyz term. It also omits degree 2 terms of the form Ax^2 , By^2 or Cz^2 We can write it as

$$f(x, y, z) = Axyz + Bxy + Cxz + Dyz + Ex + Fy + Gz + H \quad (\#\#\#)$$

which has a kind of saddle property.

Exercises

26. Show that the equation for trilinear interpolations (***) can always be rewritten like (\#\#\#) by computing A, B, C, D, E, F, G, H from $w_{000} - w_{111}$.
27. Use (***) to compute $f(x, y, z)$ when $w_{000} = 0, w_{001} = w_{010} = w_{100} = 1, w_{011} = w_{101} = w_{110} = 2, w_{111} = 3$.
28. Use (***) to compute $f(x, y, z)$ when $w_{000} = 1, w_{001} = w_{010} = w_{100} = w_{011} = w_{101} = w_{110} = w_{111} = 0$.
29. Show the equation in (\#\#\#) can be rewritten like

$$a(x-b)(y-c)(z-d) + ex + fy + gz + h \quad (\$\$\$)$$

by computing a, b, c, d, e, f, g, h from $A - H$.

10 Appendix A: A matrix game with a pure strategy solution

If the two-person zero-sum game is given by the payoff matrix

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

and we checkmark the largest entry in each column (on the left) and the smallest in each row (on the right)

$$A = \begin{bmatrix} \checkmark 5 & \checkmark 3 \checkmark \\ 3 & 2 \checkmark \end{bmatrix}$$

We see Rose plays $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and Colin plays $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and the payoff is 3. This is a game theory saddle. Alternately a movement diagram tells the story.

$$A = \begin{bmatrix} 5 & \rightarrow & 3 \\ \uparrow & & \uparrow \\ 3 & \rightarrow & 2 \end{bmatrix}$$

Colin prefers 3 to 5 on the top row and 2 to 3 on the bottom row. Rose prefers 5 to 3 on the left column and 3 to 2 on the right column. The upper right 3 has both arrows pointing towards it.

11 Appendix B: A matrix game with a mixed strategy solution

If the two-person zero-sum game is given by the payoff matrix

$$A = \begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}$$

And R plays the strategy $x = \begin{bmatrix} 1-t \\ t \end{bmatrix}$ and C plays $y = \begin{bmatrix} 1-s \\ s \end{bmatrix}$ then the expected payout is given by

$$\begin{aligned} E(x, y) &= x^T A y = 5(1-t)(1-s) + 2(1-s)t + 3s(1-t) + 4st \\ &= 5 - 5t - 5s + 5st + 2t - 2st + 3s - 3st + 4st \\ &= 5 - 3t - 2s + 4st \\ &= 4(st - s/2 - 3t/4) + 5 \\ &= 4(s - 3/4)(t - 1/2) - 3/2 + 5 \\ &= 4(s - 3/4)(t - 1/2) + 7/2 \end{aligned}$$

The equation is a hyperbolic paraboloid $z = xy$ centered at $(1/2, 3/4)$, scaled by 4 and raised by $7/2$ in the region of interest.

From this form of the payoff function we can read the value of the game is $7/2$ and the optimal strategies $\hat{x} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ and $\hat{y} = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}$.