How to show $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$

For maa4402

April 13, 2017

Our goal is to compute the improper integral

$$\int_0^\infty \frac{\sin x}{x}\, dx$$

We start by doing what seems (at first) to be an unrelated integral. Let C, (see Figure 1) be the closed path formed in four parts: The x-axis from $-R$ to $-r$; the curve $C_r : z = r\exp(i\theta),\ 0 \le \theta \le \pi$ going backwards; the x-axis from r to R; and the curve $C_R : z = R\exp(i\theta),\ 0 \le \theta \le \pi$. Let

$$f(z) = \frac{e^{iz}}{z}$$

and then

$$\int_C f(z)\, dz = \int_{-R}^{-r} f(x)\, dx - \int_{C_r} f(z)\, dz + \int_r^R f(x)\, dx + \int_{C_R} f(z) dz.$$

And since $f(z)$ is analytic everwhere but $z = 0$, the curve C has no singularities of f inside and hence

$$\int_C f(z)\, dz = 0.$$

Think of $r \to 0$ and $R \to \infty$. The curve C_r goes half way around the singularity at $z = 0$. We tackle the C_r curve first, using the following Lemma.

Lemma. *If $g(z)$ is analytic in $0 < |z| < R$ with a simple pole at $z = 0$, then*

$$\lim_{r\to 0} \int_{C_r} g(z)\, dz = \pi i \operatorname*{Res}_{z=0} g(z)$$

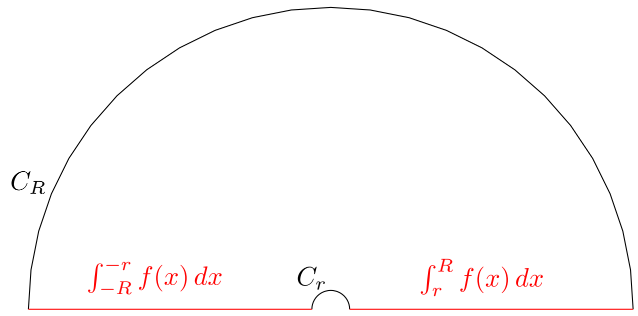


Figure 1: The four pieces of the closed contour C.

Proof. If $f(z)$ is analytic in $|z| < R$ with anti-derivative $F(z)$, then

$$\int_{C_r} f(z)\,dz = F(-r) - F(r).$$

Since $F(z)$ is continous at $z = 0$ as $r \to 0$, $F(-r) - F(r) \to F(0) - F(0) = 0$ and hence

$$\lim_{r\to 0} \int_{C_r} f(z)\,dz = 0.$$

Computing for a pole

$$\int_{C_r} \frac{1}{z}\,dz = \operatorname{Log} z\Big|_r^{-r} = \operatorname{Log}(-r) - \operatorname{Log} r = \ln r + \pi i - \ln r = \pi i$$

We can write $g(z) = \frac{B}{z} + f(z)$ and so

$$\lim_{r\to 0} \int_{C_r} g(z)\,dz = B\pi i + 0$$

$\square$

Note there are problem if $g(z)$ has a higher order pole.

$$\int_{C_r} \frac{1}{z^2}\,dz = \left.\frac{-1}{z}\right|_r^{-r} = \frac{1}{r} + \frac{1}{r} = \frac{2}{r}.$$

Which goes to ∞ as $r \to 0$.

The residue of $f(z)$ at $z = 0$ is $\exp(i0) = 1$ hence

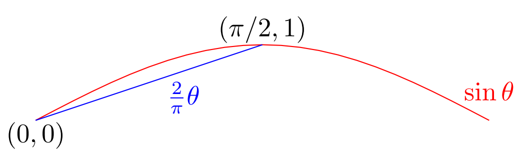
$$\lim_{r\to 0} \int_{C_r} f(z)\,dz = \pi i$$

Next we tackle C_R, we need to know some lemmas.

Lemma.

$$\textit{For } 0 \le \theta \le \frac{\pi}{2},\ \frac{2}{\pi}\theta \le \sin\theta$$

Proof.



$\square$

Note for $0 \le \theta \le \pi$:

$$\begin{aligned}
\exp(iRe^{i\theta}) &= \exp(iR(\cos\theta + i\sin\theta)) \\
&= \exp(iR\cos\theta)\exp(-R\sin\theta) \\
\left|\exp(iRe^{i\theta})\right| &= |\exp(iR\cos\theta)|\exp(-R\sin\theta) \\
&= \exp(-R\sin\theta) \\
&\le \exp(-R\frac{2}{\pi}\theta) \quad (\text{when } \theta \le \frac{\pi}{2})
\end{aligned}$$

$$\begin{aligned}
\left|\int_{C_R} f(z)\,dz\right| &\le \int_{C_R} \frac{|\exp(iz)|}{|z|}\,|dz| \\
&\le \int_0^\pi \frac{|\exp iRe^{i\theta}|}{R}|Rie^{i\theta}|\,d\theta \\
&\le \int_0^\pi |\exp iRe^{i\theta}|\,d\theta \\
&\le 2\int_0^{\pi/2} \exp(-R\frac{2}{\pi}\theta)\,d\theta \\
&= 2\frac{\exp(-R\frac{2}{\pi}\theta)}{-R\frac{2}{\pi}}\Bigg|_0^{\pi/2} \\
&= \frac{\pi}{R}(-\exp(-R)+1) \\
&\le \frac{\pi}{R} \to 0 \text{ as } R \to \infty.
\end{aligned}$$

Therefore

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$$

Finally on the x-axis, $z = x + i0$ and

$$f(z) = f(x) = \frac{e^{ix}}{x} = \frac{\cos x}{x} + i\frac{\sin x}{x}.$$

Because $\cos x/x$ is an odd function:

$$\int_{-R}^{-r} \frac{\cos x}{x}\,dx + \int_r^R \frac{\cos x}{x}\,dx = 0$$

Because $\sin x/x$ is an even function:

$$\int_{-R}^{-r} \frac{\sin x}{x}\,dx + \int_r^R \frac{\sin x}{x}\,dx = 2\int_r^R \frac{\sin x}{x}\,dx$$

$$\int_{-R}^{-r} f(x)\,dx + \int_r^R f(x)\,dx = 2i\int_r^R \frac{\sin x}{x}\,dx$$

Adding all the pieces together, as $R \to \infty$:

$$0 = 2i\int_0^\infty \frac{\sin x}{x}\,dx - \pi i$$

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

Summary of the steps:

1. Is our function even or odd or neither? Since both $\sin x$ and x are odd functions, their quotient is even and we can use
$$\int_{-\infty}^\infty \frac{\sin x}{x}\,dx = 2\int_0^\infty \frac{\sin x}{x}\,dx$$

2. Use principal values. The improper integral $\int_{-1}^{1} 1/x\,dx$ diverges in Calculus 2, since

$$\lim_{\delta\to 0^+}\int_{-1}^{-\delta}\frac{1}{x}\,dx + \lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{1}\frac{1}{x}\,dx$$

fails to exist. The principal value defined by

$$\lim_{r\to 0^+}(\int_{-1}^{-r}\frac{1}{x}\,dx \int_{r}^{1}\frac{1}{x}\,dx) = 0$$

does exist. We didn't explicitly use this step for our $f(z)$. It was hidden in the computation of $\int_{-R}^{-r}\cos x/x\,dx + \int_{r}^{R}\cos x/x\,dx = 0$. Neither improper integral exists in the Calculus 2 sense.

3. Find a nice function $f(z)$ that may have isolated singularities but is otherwise analytic (at least in the upper half plane). So that

$$\frac{\sin x}{x} = \Re(f(x+i0)).$$

For $\sin x/x$ we can use

$$\begin{aligned}
f(z) &= \frac{-i\exp(iz)}{z}\\
f(x+iy) &= \frac{-i\exp(-y+ix)}{x+iy}\\
&= \frac{-i\exp(-y)(\cos(x)+i\sin(x))}{x+iy}\\
&= \frac{\exp(-y)(\sin(x)-i\cos(x))}{x+iy}\\
f(x+i0) &= \frac{\exp(0)(\sin(x)-i\cos(x))}{x+i0}\\
f(x+i0) &= \frac{\sin(x)}{x} - i\frac{\cos(x)}{x}\\
\Re f(x+i0) &= \frac{\sin(x)}{x}
\end{aligned}$$

4. Let C_R be $\phi(\theta) = Re^{i\theta}$ for $0\le\theta\le\pi$ the top half of the circle of radius R centered at the origin. Often it is possible to show

$$\lim_{R\to 0}\int_{C_R} f(z)\,dx \to 0$$

5. Compute the integral using Cauchy Residue theorem.

6. Estimate the integral over C_R and show it goes to zero as $R\to\infty$.

7. Put the pieces together. Usually

$$\int_{-R}^{R}\Re f(x+i0)\,dx = \Re\int_{-R}^{R} f(z)\,dz + \int_{C_R} f(z)\,dz = \Re(2\pi i\sum \text{Residues})$$