Complex Homework Fall 2019

Based on Brown and Churchill 7th Edition

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Contents

These are problems will be due both daily and at the end of classes. This PDF file was created on August 5, 2019.

1 hw1, Complex Arithmetic, Conjugates, Polar Form

1. (BC3.1) Reduce each of these 3 expressions to a real number

$$\frac{1+2i}{3-4i}+\frac{2-i}{5i} \qquad \frac{5i}{(1-i)(2-i)(3-i)} \qquad \text{and} \qquad (1-i)^4$$

2. (BC4.1) In each case locate z_1+z_2 and z_1-z_2 vectorially
   $z_1=2i,\ z_2=\frac{2}{3}-i$ $\qquad z_1=(-\sqrt{3},0),\ z_2=(\sqrt{3},0)$
   $z_1=(-3,1),\ z_2=(1,4)$ $\qquad z_1=x_1+iy_1,\ z_2=x_1-iy_1$

3. (BC4.4) Sketch the set of points determined by each equation

$$|z-1+i|=1 \qquad |z+i|\le 3 \qquad \text{and} \qquad |z+4i|\ge 4$$

4. (BC5.3,4) Verify $\overline{z_1-z_2}=\overline{z_1}-\overline{z_2}$, $\overline{z_1z_2}=\overline{z}_1\overline{z}_2$, $\overline{z_1z_2z_3}=\overline{z}_1\overline{z}_2\overline{z}_3$ and $\overline{z^4}=\overline{z}^4$.

5. (BC5.5) Verify

$$\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}\ (z_2\neq 0)$$

6. (BC5.15) Show that the hyperbola $x^2-y^2=1$ can be written $z^2+\overline{z}^2=2$

7. (BC7.1) Find the principal argument $\operatorname{Arg} z$ for both

$$z=\frac{i}{-2-2i} \text{ and } z=(\sqrt{3}-i)^6$$

8. (BC7.2) Show $|e^{i\theta}|=1$ and $\overline{e^{i\theta}}=e^{-i\theta}$

9. (BC7.15) Use de Moivre's formula to derive the following trig identities.

$$\cos 3\theta=\cos^3\theta-3\cos\theta\sin^2\theta=4\cos^3\theta-3\cos\theta$$

$$\sin 3\theta=3\cos^2\theta\sin\theta-\sin^3\theta=3\sin\theta-4\sin^3\theta$$

2 hw2 nth roots, Domains, Functions

1. (BC7.7) Show if $\Re z_1>0$ and $\Re z_2>0$ then $\operatorname{Arg}(z_1z_2)=\operatorname{Arg} z_1+\operatorname{Arg} z_2$
2. (BC9.1) Find the square roots of $2i$ and $1-i\sqrt{3}$ expressed in rectangular form
3. (BC9.3) Find all of the roots in rectangle coordinates of $(-1)^{1/3}$ and $8^{1/6}$.

4. (BC9.6) Find the 4 roots of $p(z) = z^4 + 4 = 0$ and use them to factor $p(z)$ into quadratic factors with real coefficients.

5. (BC10.1-3) Sketch the 6 sets and determine which are domains, which are bounded, which are neither open nor closed:

   $|z - 2 + i| \le 1$ $\quad |2z + 3| > 4 \quad \Im z > 1$

   $\Im z = 1 \quad 0 \le \arg z \le \pi/4\,(z \ne 0) \quad |z - 4| \le |z|$

6. (BC10.4) Find the closure of the 4 sets:

$$-\pi < \arg z < \pi\,(z \ne 0) \qquad |\Re z| < |z| \qquad \Re(\frac{1}{z}) \le \frac{1}{2} \qquad \text{and} \qquad \Re(z^2) > 0$$

7. (BC11.1) For each function, describe the domain that is understood:

$$f(z) = \frac{1}{z^2+1} \qquad f(z) = \operatorname{Arg}(\frac{1}{z}) \qquad f(z) = \frac{z}{z+\overline{z}} \qquad \text{and} \qquad f(z) = \frac{1}{1-|z|^2}$$

8. (BC11.2) Write $z^3 + z + 1$ as $u(x, y) + iv(x, y)$

9. (BC11.3) Write and simplify $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$ in terms of z using $x = (z + \overline{z})/2$ and $y = (z - \overline{z})/2i$

10. (BC11.4) Write $f(z) = z + 1/z\,(z \ne 0)$ in the form $u(r, \theta) + iv(r, \theta)$

3 hw3 Images, Transformations

1. (BC13.1) Find a domain in the z-plane whose image under the transformation $w = z^2$ is the square domain in the w-plane bounded by the lines $u = 1, u = 2, v = 1, v = 2$

2. (BC13.3) Sketch the region onto which the sector $r \le 1, 0 \le \theta \le \pi/4$ is mapped by the 3 transformations $w = z^2, w = z^3$, and $w = z^4$

3. (BC13.4) Show that lines $ay = x\,(a \ne 0)$ are mapped onto the spirals $\rho = \exp(a\theta)$ under the transformation $w = \exp z$, where $w = \rho \exp(i\phi)$

4. (BC13.7) Find the image of the semi-infinite strip $x \ge 0, 0 \le y \le \pi$ under the transformation $w = \exp z$. Label the corresponding portions of the boundaries.

5. (BC13.8) Graphically indicate the vector fields represented by $w = iz$ and $w = z/|z|$

4 hw4 Limits

1. (BC17.3) Find the limits. n is a positive integer, $P(z)$ and $Q(z)$ are polynomials with $Q(z_0) \ne 0$

$$\lim_{z\to z_0} \frac{1}{z^n}\,(z_0 \ne 0) \qquad \lim_{z\to i} \frac{iz^3-1}{z+i} \qquad \text{and} \qquad \lim_{z\to z_0} \frac{P(z)}{Q(z)}$$

2. (BC17.5) Show that the following limit does not exist

$$\lim_{z\to 0} (\frac{z}{\overline{z}})^2$$

3. (BC17.10) Use a theorem to show:

$$\lim_{z\to\infty} \frac{4z^2}{(z-1)^2} = 4 \qquad \lim_{z\to 1} \frac{1}{(z-1)^3} = \infty \qquad \text{and} \qquad \lim_{z\to\infty} \frac{z^2+1}{z-1} = \infty$$

4. (BC17.11) Suppose $ad - bc \neq 0$ and let:

$$T(z) = \frac{az+b}{cz+d}$$

Use a theorem to show

$$\lim_{z\to\infty} T(z) = \infty \,(\text{if } c = 0) \qquad \lim_{z\to\infty} T(z) = \frac{a}{c}\,(\text{if } c \neq 0) \quad \text{and} \quad \lim_{z\to -d/c} T(z) = \infty \,(\text{if } c \neq 0)$$

5 hw5 Unbounded

1. (BC17.13) Show that a set S is unbounded if and only if every neighborhood of the point at infinity contains at least one point of S.

6 hw6 Derivatives, Cauchy-Riemann

1. (BC19.1) Find $f'(z)$ when

$$f(z) = 3z^2 - 2z + 4 \quad f(z) = (1-4z^2)^3 \quad f(z) = \frac{z-1}{2z+1}\ (z \neq -\frac{1}{2}) \quad \text{and} \quad f(z) = \frac{(1+z^2)^4}{z^2}\ (z \neq 0)$$

2. (BC19.2) Show if $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ then $P'(z) = a_1 + 2a_2 z + \cdots + n a_z z^{n-1}$ and hence

$$a_0 = P(0), \quad a_1 = \frac{P'(0)}{1!}, \quad a_2 = \frac{P''(0)}{2!}, \quad \ldots \quad a_n = \frac{P^{(n)}(0)}{n!}$$

3. (BC19.9) Let f denote the function whose values are

$$f(z) = \begin{cases} \overline{z}^2/z & \text{when } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}$$

Show that if $z = 0$, then $\Delta w/\Delta z = 1$ at each nonzero point on the real and imaginary axes in the Δz or $\Delta x \Delta y$-plane. Then show then $\Delta w/\Delta z = -1$ at each nonzero point along the line $y = x$. Conclude that $f'(0)$ does not exist.

4. (BC22.6) Let f denote the function above. Show that the Cauchy-Riemann equations are satisfied at the origin $z = (0, 0)$

5. (BC22.1) Use a theorem to show that $f'(z)$ does not exist at any point for each function:

$$f(z) = \overline{z} \qquad f(z) = z - \overline{z} \qquad f(z) = 2x + ixy^2 \qquad \text{and} \qquad f(x) = e^x e^{-iy}$$

6. (BC22.2) Use a theorem to show that $f'(z)$ and its derivative $f''(z)$ exist everywhere and find $f''(z)$.

$$f(z) = iz + 2 \qquad f(z) = e^{-x} e^{-iy} \qquad f(z) = z^3 \quad \text{and} \quad f(z) = \cos x \cosh y - i \sin x \sinh y$$

7. Extra Credit (BC22.10) Recall $z = x + iy$ implies $x = (z + \overline{z})/2$ and $y = (z - \overline{z})/2i$. Use the formal chain rule to show
$$\frac{\partial F}{\partial \overline{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y})$$
Define the operator
$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$
and apply it to $u(x, y)+iv(x, y)$ to obtain the complex form of the Cauchy-Reimann equations $\partial f/\partial \overline{z} = 0$.

7 hw7 Exp and Log

1. (BC28.1) Show that $\exp(2 \pm 3\pi i) = -e^2$, $\exp((2 + \pi i)/4) = (1 + i)\sqrt{e/2}$ and $\exp(z + \pi i) = -\exp z$.
2. (BC28.2) State why the function $2z^2 - 3 - ze^z + e^{-z}$ is entire.
3. (BC28.3) Show $f(z) = \exp \overline{z}$ is not analytic anywhere.
4. (BC28.7) Prove $|\exp(-2z)| < 1$ if and only if $\Re z > 0$.
5. (BC28.8) Find all values of z such that $e^z = -2$, or $e^z = 1 + \sqrt{3}i$ or $\exp(2z - 1) = 1$
6. (BC28.10) Show that if e^z is real, then $\Im z = n\pi \qquad (n = 0, \pm 1, \pm 2, \dots)$. If e^z is pure imaginary, what restriction is placed on z?
7. (BC30.1) Show that $\mathrm{Log}(-ei) = 1 - \frac{\pi}{2}i$ and $\mathrm{Log}(1 - i) = \frac{1}{2}\ln 2 - \frac{\pi}{4}i$.

8 hw8 Log and log

1. (BC30.2) Verify for $n = 0, \pm 1, \pm 2, \dots$:
$$\log e = 1 + 2n\pi i \qquad \log i = (2n + \frac{1}{2})\pi i \quad \text{and} \quad \log(-1 + \sqrt{3}i) = \ln 2 + 2(n + \frac{1}{3})\pi i$$
.
2. (BC30.3) Show that $\mathrm{Log}(1 + i)^2 = 2\,\mathrm{Log}(1 + i)$ and $\mathrm{Log}(-1 + i)^2 \neq 2\,\mathrm{Log}(-1 + i)$.
3. (BC30.5) Show that the set of values of $\log(i^{1/2})$ is $\{(n + \frac{1}{4})\pi i : n = 0, \pm 1, \pm 2, \dots\}$ and that the same is true of $(1/2)\log i$.
4. (BC30.6) Given that the branch $\log z = \ln r + i\theta\,(r > 0, \alpha < \theta < \alpha + 2\pi)$ of the logarithmic function is analytic at each point z in the stated domain, obtain its derivative by differentiating each side of the identity $\exp(\log z) = z$ and using the chain rule.
5. (BC30.7) Find all the roots of the equation $\log z = i\pi/2$.
6. (BC30.9) Show that $\mathrm{Log}(z - i)$ is analytic everywhere except on the half line $y = 1\,(x \leq 0)$. Show
$$\frac{\mathrm{Log}(z + 4)}{z^2 + i}$$
is analytic everywhere except at the points $\pm(1 - i)/\sqrt{2}$ and on the portion $x \leq -4$ of the real axis.

9 hw9 Principal values, Integrals over a Real Variable

1. (BC31.1) Show if $\Re z_1 > 0$ and $\Re z_2 > 0$ then $\mathrm{Log}(z_1 z_2) = \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2$.

2. (BC31.2) Show that for any two complex numbers z_1 and z_2, $\mathrm{Log}(z_1 z_2) = \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2 + 2N\pi i$ where N has one of the values $0, \pm 1$.

3. (BC32.1) Show that when $n = 0, \pm 1, \pm 2 \ldots$

$$(1+i)^i = \exp(-\frac{\pi}{4} + 2n\pi) \exp(\frac{i}{2} \ln 2) \qquad \text{and} \qquad (-1)^{1/\pi} = e^{(2n+1)i}$$

4. (BC32.2) Find the principal values of each expression:

$$i^i \qquad [\frac{e}{2}(-1 - \sqrt{3}i)]^{3\pi i} \qquad \text{and} \qquad (1-i)^{4i}$$

5. (BC32.5) Show that the principal n-th root of a nonzero complex number z_0 is the same as the principal value of $z_0^{1/n}$ that was previously defined.

6. (BC32.8) Let c, d, z be complex numbers with $z \neq 0$. Prove that if all the powers involved are principal values, then

$$\frac{1}{z^c} = z^{-c} \qquad (z^c)^n = z^{cn}\ (n = 1, 2, \ldots) \qquad z^c z^d = z^{c+d} \qquad \text{and} \qquad \frac{z^c}{z^d} = z^{c-d}$$

7. (BC37.2) Evaluate

$$\int_1^2 (\frac{1}{t} - i)^2\, dt \qquad \int_0^{\pi/6} e^{i2t}\, dt \qquad \text{and} \qquad \int_0^{\infty} e^{-zt}\, dt\ (\Re z > 0)$$

.

8. (BC37.5) Let $w(t)$ be a continuous complex-valued funtion of t defined on an interval $a \leq t \leq b$. By considering the special case $w(t) = e^{it}$ on the interval $0 \leq t \leq 2\pi$, show that it is not always true that there is a number c in the interval $a < t < b$ such that

$$\int_a^b w(t)\, dt = w(c)(b-a)$$

10 hw10 Contour Integrals

1. (BC38.2) Let C denote the right-hand half of the circle $|z| = 2$, in the counterclockwise direction and note that two parametric representations for C are

$$z = z(\theta) = 2e^{i\theta} \quad (-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2})$$

and

$$z = Z(y) = \sqrt{4 - y^2} + iy \quad (-2 \leq y \leq 2)$$

Verify that $Z(y) = z[\phi(y)]$, where

$$\phi(y) = \arctan \frac{y}{\sqrt{4 - y^2}} \qquad (-\frac{\pi}{2} \leq \arctan t \leq \frac{\pi}{2})$$

Also, show that this function ϕ has a positive derivative, as required in the conditions following (9) Sec 38.

2. (BC40.1,2,3,5,6) Evaluate

$$\int_C f(z)\,dz$$

for the given $f(z)$ and contour C

$f(z) = (z+2)/z$	C is $z = 2e^{i\theta}\,(0 \le \theta \le \pi)$
$f(z) = (z+2)/z$	C is $z = 2e^{i\theta}\,(\pi \le \theta \le 2\pi)$
$f(z) = (z+2)/z$	C is $z = 2e^{i\theta}\,(0 \le \theta \le 2\pi)$
$f(z) = z+1$	C is $z = 1 + e^{i\theta}\,(\pi \le \theta \le 2\pi)$
$f(z) = z+1$	C is $z = t\,(0 \le t \le 2)$
$f(z) = \pi\exp(\pi\overline{z})$	C is square from $0, 1, 1+i, i$
$f(z) = 1$	C is arbitrary curve from z_1 to z_2
$f(z) = z^{-1+i}$	C is $\|z\| = 1$ positively oriented
	use branch $\exp[(-1+i)\log z]\,(\|z\| > 0, 0 < \arg z < 2\pi)$

3. (BC40.10) Let C_0 denote the circle $|z - z_0| = R$ taken counterclockwise. Use the parametric representation $z = z_0 + Re^{i\theta}\,(-\pi \le \theta \le \pi)$ for C_0 to derive the following integration formula's:

$$\int_{C_0} \frac{dz}{z - z_0} = 2\pi i \qquad \text{and} \qquad \int_{C_0} (z - z_0)^{n-1}\,dz = 0\,(n = \pm 1, \pm 2, \dots)$$

11 hw11 More on Contour Integrals

1. (BC41.4) Let C_R denote the upper half of the circle $|z| = R\,(R > 2)$, taken in the counterclockwise direction. Show that

$$\left|\int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4}\,dz\right| \le \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)}$$

2. (BC43.1) Use an antiderivative to show that, for every contour C extending from a point z_1 to a point z_2,

$$\int_C z^n\,dz = \frac{1}{n+1}(z_2^{n+1} - z_1^{n+1})\,(n = 0, 1, \dots)$$

3. (BC43.2) By finding an antiderivative, evaluate each of these integrals, where the path is any contour between the indicated limits of integration.

$$\int_i^{i/2} e^{\pi z}\,dz \qquad \int_0^{\pi+2i} \cos(\frac{z}{2})\,dz \qquad \text{and} \qquad \int_1^3 (z-2)^3\,dz$$

12 hw12 Path independence

1. (BC43.3) Use a theorem to show

$$\int_{C_0} (z - z_0)^{n-1}\,dz = 0\,(n = \pm 1, \pm 2, \dots)$$

when C_0 is any closed contour which does not pass through the point z_0.

2. (BC43.4) Let C_1, (resp. C_2), be any contour from $z = -3$ to $z = 3$ that except for its end points, lies above (resp. below) the x-axis. Find an antiderivative $F_2(z)$ of the branch $f_2(z)$ of

$$z^{1/2} = \sqrt{r}e^{i\theta/2} \quad (r > 0, \frac{\pi}{2} < \theta < \frac{5\pi}{2})$$

to show that the integral

$$\int_{C_2} z^{1/2}\, dz$$

has value $2\sqrt{3}(-1+i)$. Note that the value of the integral of the function

$$z^{1/2} = \sqrt{r}e^{i\theta/2}$$

around the closed contour $C_2 - C_1$ in that example is, therefore $-4\sqrt{3}$ given that

$$\int_{C_1} z^{1/2}\, dz = 2\sqrt{3}(1+i)$$

. (Lots of parts from example 43.4.)

13 hw13 Cauchy Goursat

1. (BC46.1) Apply the Cauchy-Goursat theorem to show that

$$\int_C f(z)\, dz = 0$$

when the contour C is the circle $|z| = 1$, in either direction and when

$$f(z) = \frac{z^2}{z-3} \qquad f(z) = ze^{-z} \qquad f(z) = \frac{1}{z^2+2z+2}$$
$$f(z) = \operatorname{sech} z \qquad f(z) = \tan z \qquad f(z) = \operatorname{Log}(z+2)$$

2. (BC46.2) Let C_1 be the positively oriented circle $|z| = 4$ and let C_2 be the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1, y = \pm 1$. Point out why

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz$$

when

$$f(z) = \frac{1}{3z^2+1} \qquad f(z) = \frac{z+2}{\sin(z/2)} \qquad \text{and} \qquad f(z) = \frac{z}{1-e^z}$$

3. (BC46.3) If C is the boundary of the rectangle $0 \le x \le 3, 0 \le y \le 2$, described in the positive sense, then

$$\int_C (z-2-i)^{n-1} = 2\pi i \text{ when } n = 0 \text{ and } 0 \text{ when } n = \pm 1, \pm 2, \ldots$$

4. (BC46.4) Extra Credit ????

14 hw14 Applications of Cauchy Integral Formula

1. (BC48.1abc) Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2, y = \pm 2$. Evaluate the integrals

$$\int_C \frac{e^{-z}\, dz}{z-(\pi i/2)} \qquad \int_C \frac{\cos z\, dz}{z(z^2+8)} \qquad \text{and} \qquad \int_C \frac{z\, dz}{2z+1}$$

2. (BC48.2) Find the integral of $g(z)$ around the circle $|z - i| = 2$ in the positive sense when $g(z) = 1/(z^2+4)$ and when $g(z) = 1/(z^2+4)^2$.

3. (BC48.3) Let C be the circle $|z| = 3$ decribed in the positive sense. Show that if

$$g(w) = \int_C \frac{2z^2 - z - 2}{z - w}\, dz \qquad (|w| \neq 3)$$

then $g(2) = 8\pi i$. What is the value of $g(w)$ when $|w| > 3$?

4. (BC48.7) Let C be the unit circle $z = e^{i\theta}$ $(-\pi \leq \theta \leq \pi)$. First show that for any real constant a,

$$\int_C \frac{e^{az}}{z}\, dz = 2\pi i$$

Then write this integral in terms of θ to derive the integration formula

$$\int_0^\pi e^{a\cos\theta} \cos(a \sin\theta)\, d\theta = \pi$$

5. (BC48.6) Extra Credit ???? Let f denote a function that is *continuous* on a simple closed contour C. Prove the function

$$g(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)\, d\xi}{\xi - z}$$

is analytic as each point z interior to C and and that

$$g'(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)\, d\xi}{(\xi - z)^2}$$

at such a point.

15 hw15 Liouville

1. (BC50.1) Let f be an entire function such that $|f(z)| \leq A|z|$ for all z, where A is a fixed positive number. Show that $f(z) = a_1 z$, where a_1 is a complex constant. [Hint: use Cauchy's inequality to show $f''(z)$ is zero.]

2. (BC50.1) Suppose $f(z)$ is entire and that the harmonic function $u(x, y) = \Re f(z)$ has an upper bound u_0: that is, $u(x, y) \leq u_0$ for all points (x, y) in the xy-plane. Show that $u(x, y)$ must be constant throughout the plane. [Hint: use Liouville's theorem on $\exp(f(z))$.]

3. (BC50.4,5) Let a function f be continuous in a closed bounded region R, and let it be analytic and not constant throughout the interior of R. Assuming $f(z) \neq 0$ anywhere in R, prove that $|f(z)|$ has a *minimum value* m in R which occurs on the boundary of R and never in the interior. [Hint: look at $1/f(z)$.]

   Use the function $f(z) = z$ to show that the condition $f(z) \neq 0$ anywhere is necessary for this conclusion.

16 hw16 Series

1. (BC52.6) Show if $\sum_{n=1}^\infty z_n = S$, then $\sum_{n=1}^\infty \overline{z}_n = \overline{S}$.

2. (BC52.7) Show for any complex number c Show if $\sum_{n=1}^\infty z_n = S$, then $\sum_{n=1}^\infty c z_n = cS$.

3. (BC52.8) Show if $\sum_{n=1}^\infty z_n = S$ and $\sum_{n=1}^\infty w_n = T$, then $\sum_{n=1}^\infty (z_n + w_n) = S + T$.

17 hw17 Taylor Series

1. (BC54.2) Obtain the Taylor

$$e^z = e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!} \qquad (|z-1|<\infty)$$

two ways. First using $f^{(n)}(1)$ and second by using $e^z = ee^{z-1}$.

2. (BC54.3) Find the Maclaurin series expansion for the function

$$f(z) = \frac{z}{z^4+9} = \frac{z}{9}\cdot\frac{1}{1+z^4/9}$$

3. (BC54.5) Derive the Maclaurin series for $\cos z$ by showing $f^{(2n)}(0) = (-1)^n$ and $f^{(2n+1)}(0) = 0$ and by using $\cos z = (e^{iz}+e^{-iz})/2$.

4. (BC54.11) Show when $z \neq 0$,

$$\frac{e^z}{z^2} = \frac{1}{z^2}+\frac{1}{z}+\frac{1}{2!}+\frac{z}{3!}+\frac{z^2}{4!}+\cdots$$
$$\frac{\sin(z^2)}{z^4} = \frac{1}{z^2}-\frac{z^2}{3!}+\frac{z^6}{5!}-\frac{z^{10}}{7!}+\cdots$$

5. (BC54.13) Show that when $0<|z|<4$,

$$\frac{1}{4z-z^2} = \frac{1}{4z}+\sum_{n=0}^{\infty}\frac{z^n}{4^{n+2}}$$

18 hw18 Laurent Series

1. (BC56.1) Find the Laurent series that represents the function $f(z) = z^2\sin(1/z^2)$ in the domain $0<z<\infty$.

2. (BC56.2) Derive the Laurent series representation

$$\frac{e^z}{(z+1)^2} = \frac{1}{e}\left[\sum_{n=0}^{\infty}\frac{(z+1)^n}{(n+2)!}+\frac{1}{z+1}+\frac{1}{(z+1)^2}\right]$$

3. (BC56.3) Find a representation for the function

$$f(z) = \frac{1}{1+z} = \frac{1}{z}\cdot\frac{1}{1+(1/z)}$$

in negative powers of z that is valid for $1<|z|<\infty$.

4. (BC56.4) Give two Laurent series expansions in powers of z for the function $f(z) = 1/[z^2(1-z)]$ and specify the regions in which the expansions are valid. [Hint: about 0 and ∞]

5. (BC56.5) Represent the function

$$f(z) = \frac{z+1}{z-1}$$

by both its Maclaurin series (stating where it is valid) and by a Laurent series in the domain $1<|z|<\infty$

6. (BC56.6) Show that when $0<|z-1|<2$,

$$\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}}-\frac{1}{2(z-1)}$$

19 hw19 Derivative of Series, Substituting, Poles, Residues

1. (BC60.1) By differentiating the Maclaurin series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1)$$

obtain the expressions

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n \qquad (|z| < 1)$$

and

$$\frac{2}{(1-z)^3} = \sum_{n=0}^{\infty} (n+1)(n+2) z^n \qquad (|z| < 1)$$

2. (BC60.2) By substituting $1/(1-z)$ for z in the expansion

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n \qquad (|z| < 1)$$

found above, derive the Laurent series representation

$$\frac{1}{z^2} = \sum_{n=2}^{\infty} \frac{(-1)^n (n-1)}{(z-1)^n} \qquad (1 < |z-1| < \infty)$$

3. (BC60.3) Find the Taylor series for the function

$$\frac{1}{z} = \frac{1}{2+(z-2)} = \frac{1}{2} \cdot \frac{1}{1+(z-2)/2}$$

about the point $z_0 = 2$. Then by differentiating that series term by term, show that

$$\frac{1}{z^2} = \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n (n+1) (\frac{z-2}{2})^n \qquad (|z-2| < 2)$$

4. (BC61.1) Use multiplication of series to show that

$$\frac{e^z}{z(z^2+1)} = \frac{1}{z} + 1 - \frac{1}{2} z - \frac{5}{6} z^2 + \cdots \qquad (0 < |z| < 1)$$

5. (BC61.3) Use division to obtain the Laurent series representation

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \frac{1}{12} z - \frac{1}{720} z^3 + \cdots \qquad (0 < |z| < 2\pi)$$

6. (BC64.1) Find the residue at $z = 0$ of the functions

$$\frac{1}{z+z^2} \qquad z\cos(\frac{1}{z}) \qquad \frac{z-\sin z}{z} \qquad \frac{\cot z}{z^4} \qquad \text{and} \qquad \frac{\sinh z}{z^4(1-z^2)}$$

7. (BC64.2) Use Cauchy's residue theorem to evaluate the integral of each of these functions around the circle $|z| = 3$ in the positive sense:

$$\frac{\exp(-z)}{z^2} \qquad \frac{\exp(-z)}{(z-1)^2} \qquad z^2 \exp(\frac{1}{z}) \qquad \text{and} \qquad \frac{z+1}{z^2-2z}$$

8. (BC64.3) Use a theorem involving a single residue to evaluate the integral of each of these functions around the circle $|z| = 2$ in the positive sense.

$$\frac{z^5}{1-z^3} \qquad \frac{1}{1+z^2} \qquad \text{and} \qquad \frac{1}{z}$$

20 hw20 Singular points

1. (BC65.1) In each case, write the principal part of the function at its isolated singular point and determine whether that point is a pole, a removable singular point or an essential singular pont.

$$z\exp(\frac{1}{z}) \qquad \frac{z^2}{1+z} \qquad \frac{\sin z}{z} \qquad \frac{\cos z}{z} \qquad \text{and} \qquad \frac{1}{(2-z)^3}$$

2. (BC65.2) Show that the singular point of each of the following functions is a pole. Determine the order m of the pole and the corresponding residue B.

$$\frac{1-\cosh z}{z^3} \qquad \frac{1-\exp(2z)}{z^4} \qquad \text{and} \qquad \frac{\exp(2z)}{(z-1)^2}$$

3. (BC65.3) Suppose f is analytic at z_0 and write $g(z) = f(z)/(z - z_0)$. Show that:
   (a) If $f(z_0) \neq 0$, then z_0 is a simple pole of g, with residue $f(z_0)$.
   (b) If$f(z_0) = 0$, then z_0 is a removable singular point of g.

21 hw21 Residues, Poles, Order of a Pole

1. (BC65.4) Write the function

$$f(z) = \frac{8a^3z^2}{(z^2+a^2)^3} \qquad (a > 0)$$

as

$$f(z) = \frac{\phi(z)}{(z-ai)^3} \text{ where } \phi(z) = \frac{8a^3z^2}{(z+ai)^3}$$

Point out why $\phi(z)$ has a Taylor series representation about $z = ai$, and then use it to show that the principal part of f at that point is

$$\frac{\phi''(ai)/2}{z-ai} + \frac{\phi'(ai)}{(z-ai)^2} + \frac{\phi(ai)}{(z-ai)^3} = -\frac{i/2}{z-ai} - \frac{a/2}{(z-ai)^2} - \frac{a^2i}{(z-ai)^3}$$

2. (BC67.1) In each case, show that any singular point of the function is a pole. Determine the order m of the pole and find the corresponding residue B

$$\frac{z^2+2}{z-1} \qquad (\frac{z}{2z+1})^3 \qquad \text{and} \qquad \frac{\exp z}{z^2+\pi^2}$$

3. (BC67.2) Show that

$$\operatorname*{Res}_{z=-1} \frac{z^{1/4}}{z+1} = \frac{1+i}{\sqrt{2}} \qquad (|z| > 0, 0 < \arg z < 2\pi)$$

$$\operatorname*{Res}_{z=i} \frac{\operatorname{Log} z}{(z^2+1)^2} = \frac{\pi+2i}{8}$$

$$\operatorname*{Res}_{z=i} \frac{z^{1/2}}{(z^2+1)^2} = \frac{1-i}{8\sqrt{2}} \qquad (|z| > 0, 0 < \arg z < 2\pi)$$

4. (BC67.3) Find the value of the integral

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\,dz$$

taken counterclockwise around both circles $|z-2| = 2$ and $|z| = 4$

22 hw22 Computing Integrals

1. (BC67.4) Find the value of the integral
$$\int_C \frac{dz}{z^3(z+4)}$$
taken counterclockwise around both circles $|z| = 2$ and $|z+2| = 3$

2. (BC69.1) Show that the point $z = 0$ is a simple pole of the function $f(z) = \csc z = 1/\sin z$ by a theorem and by computing the Laurent series.

3. (BC69.3a) Show that
$$\operatorname*{Res}_{z=z_n}(z \sec z) = (-1)^{n+1} z_n, \text{ where } z_n = \frac{\pi}{2} + n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots$$

4. (BC69.4a) Let C denote the positively oriented circle $|z| = 2$ and evaluate the integral
$$\int_C \tan z \, dz$$

5. (BC69.5) Let C_N denote the positive oriented boundary of the square whose edges lie along the lines
$$x = \pm(N + \frac{1}{2})\pi \text{ and } y = \pm(N + \frac{1}{2})\pi$$
where N is a positive integer. Show that
$$\int_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i \left[\frac{1}{6} + 2 \sum_{n=1}^{N} \frac{(-1)^n}{n^2 \pi^2} \right]$$
then using the fact that the value of this integral tends to zero as N tends to infinity, point out how it follows that
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

23 hw23 Poles and Zeros

1. (BC69.9) Let p and q denote functions that are analytic at a point z_0 where $p(z_0) \neq 0$ and $q(z_0) = 0$. Show that if the quotient $p(z)/q(z)$ has a pole of order m at z_0, then z_0 is a zero of order m of q.

24 hw24 Cool Integrals

1. (BC72.1,2,4) Use residues to evaluate the following integrals
$$\int_0^\infty \frac{dx}{x^2+1} \qquad \int_0^\infty \frac{dx}{(x^2+1)^2} \qquad \text{and} \qquad \int_0^\infty \frac{x^2\, dx}{(x^2+1)(x^2+4)}$$

2. (BC74.1,2) Use residues to evaluate the following integrals
$$\int_{-\infty}^\infty \frac{\cos x \, dx}{(x^2+a^2)(x^2+b^2)} \quad (a > b > 0) \qquad \text{and} \qquad \int_0^\infty \frac{\cos ax \, dx}{x^2+1} \quad (a > 0)$$