Harmonic Exercises

For maa4402

December 29, 2016

For each $u(x,y)$, first show $u(x,y)$ is harmonic and then find $v(x,y)$ so that $f(z) = u(x,y) + iv(x,y)$ is an entire function.

1. $u(x,y) = x^5 - 10x^3y^2 + 5xy^4$.

   Answer First lets show that u is harmonic:

   $$u_{xx} = (5x^4 - 30x^2y^2 + 5y^4)_x = 20x^3 - 60xy^2$$

   $$u_{yy} = (-20x^3y + 20xy^3)_y = -20x^3 + 60xy^2 = -u_{xx}$$

   Cauchy-Riemann say $v_x = -u_y = 20x^3y - 20xy^3$ and $v_y = u_x = 5x^4 - 30x^2y^2 + 5y^4$. Just like in Calculus 3 this implies

   $$v = 5x^4y - 10x^2y^3 + \Theta(x)$$

   $$v = 5x^4y - 10x^2y^3 + y^5 + \Phi(y)$$

   Combining we get

   $$v = 5x^4y - 10x^2y^3 + y^5 + C$$

   Eventually, if $C = 0$, $f(z) = z^5$ since

   $$u + iv = x^5 + 5x^4iy + 10x^3(iy)^2 + 10x^2(iy)^3 + (iy)^5 = (x+iy)^5$$

2. $u(x,y) = -y$

3. $u(x,y) = xy$

4. $u(x,y) = y^2 - x^2$

5. $u(x,y) = 3xy^2 - x^3$

6. $u(x,y) = \cos xe^y$

7. $u(x,y) = \cos x \sinh y$

1. Hints
2. $\Re(iz) = -y$
3. $\Re(-iz^2/2) = xy$
4. $\Re(-z^2) = y^2 - x^2$
5. $\Re(-z^3) = 3xy^2 - x^3$
6. $\Re(\exp(-iz) = \cos x \exp y$
7. $\Re(-i \sin(z)) = \cos x \sinh y$