(b) Use the result in part (a) to find the roots of the equation $z^2 + 2z + (1 - i) = 0$.

$$Ans. (b) \left(-1 + \frac{1}{\sqrt{2}}\right) + \frac{i}{\sqrt{2}}, \quad \left(-1 - \frac{1}{\sqrt{2}}\right) - \frac{i}{\sqrt{2}}.$$

9. Let $z = re^{i\theta}$ be a nonzero complex number and n a negative integer $(n = -1, -2, \ldots)$. Then define $z^{1/n}$ by means of the equation $z^{1/n} = (z^{-1})^{1/m}$ where $m = -n$. By showing that the m values of $(z^{1/m})^{-1}$ and $(z^{-1})^{1/m}$ are the same, verify that $z^{1/n} = (z^{1/m})^{-1}$. (Compare with Exercise 7, Sec. 8.)

11. REGIONS IN THE COMPLEX PLANE

In this section, we are concerned with sets of complex numbers, or points in the z plane, and their closeness to one another. Our basic tool is the concept of an ε *neighborhood*

$$|z - z_0| < \varepsilon \tag{1}$$

of a given point z_0. It consists of all points z lying inside but not on a circle centered at z_0 and with a specified positive radius ε (Fig. 15). When the value of ε is understood or is immaterial in the discussion, the set (1) is often referred to as just a neighborhood. Occasionally, it is convenient to speak of a *deleted neighborhood*, or punctured disk,

$$0 < |z - z_0| < \varepsilon \tag{2}$$

consisting of all points z in an ε neighborhood of z_0 except for the point z_0 itself.

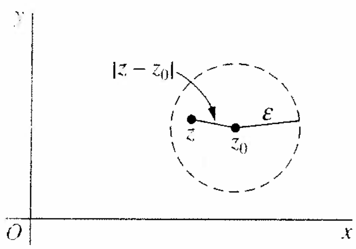

FIGURE 15

A point z_0 is said to be an *interior point* of a set S whenever there is some neighborhood of z_0 that contains only points of S; it is called an *exterior point* of S when there exists a neighborhood of it containing no points of S. If z_0 is neither of these, it is a *boundary point* of S. A boundary point is, therefore, a point all of whose neighborhoods contain at least one point in S and at least one point not in S. The totality of all boundary points is called the *boundary* of S. The circle $|z| = 1$, for instance, is the boundary of each of the sets

$$|z| < 1 \quad \text{and} \quad |z| \leq 1. \tag{3}$$

A set is *open* if it contains none of its boundary points. It is left as an exercise to show that a set is open if and only if each of its points is an interior point. A set is *closed* if it contains all of its boundary points, and the *closure* of a set S is the closed set consisting of all points in S together with the boundary of S. Note that the first of the sets (3) is open and that the second is its closure.

Some sets are, of course, neither open nor closed. For a set to be not open, there must be a boundary point that is contained in the set; and if a set is not closed, there exists a boundary point not contained in the set. Observe that the punctured disk $0 < |z| \le 1$ is neither open nor closed. The set of all complex numbers is, on the other hand, both open and closed since it has no boundary points.

An open set S is *connected* if each pair of points z_1 and z_2 in it can be joined by a *polygonal line*, consisting of a finite number of line segments joined end to end, that lies entirely in S. The open set $|z| < 1$ is connected. The annulus $1 < |z| < 2$ is, of course, open and it is also connected (see Fig. 16). A nonempty open set that is connected is called a *domain*. Note that any neighborhood is a domain. A domain together with some, none, or all of its boundary points is referred to as a *region*.

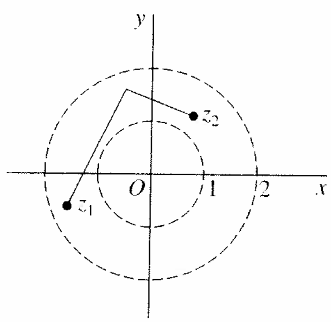

FIGURE 16

A set S is *bounded* if every point of S lies inside some circle $|z| = R$; otherwise, it is *unbounded*. Both of the sets (3) are bounded regions, and the half plane $\text{Re } z \ge 0$ is unbounded.

A point z_0 is said to be an *accumulation point* of a set S if each deleted neighborhood of z_0 contains at least one point of S. It follows that if a set S is closed, then it contains each of its accumulation points. For if an accumulation point z_0 were not in S, it would be a boundary point of S; but this contradicts the fact that a closed set contains all of its boundary points. It is left as an exercise to show that the converse is, in fact, true. Thus a set is closed if and only if it contains all of its accumulation points.

Evidently, a point z_0 is *not* an accumulation point of a set S whenever there exists some deleted neighborhood of z_0 that does not contain at least one point of S. Note that the origin is the only accumulation point of the set $z_n = i/n$ $(n = 1, 2, \ldots)$.