Extra Problems

1. Each matrix A below can be considered a linear map from $(\mathbb{R}^2, \|\cdot\|_p)$ to itself. Compute the norm when $p = 1, 2$ and ∞

   (a)
   $$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

   (b)
   $$A = \begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix}$$

   (c)
   $$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

   (d)
   $$A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

2. Suppose $1 \leq r < p, q \leq \infty$ with
   $$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$
   show for sequences (x_n) and (y_n) and functions f and g
   $$\||(x_n)(y_n)|\|_r \leq \|(x_n)\|_p \|(y_n)\|_q$$
   $$\||f(t)g(t)|\|_r \leq \|f(t)\|_p \|g(t)\|_q$$

3. Let $\alpha > 0$

   (a) For what values of p is the sequence $(1/n^\alpha) \in \ell_p$

   (b) For what values of p is the function $f(x) = 1/x^\alpha \in L_p([0,1])$

   (c) For what values of r is $L_p(\mathbb{R}) \cap L_q(\mathbb{R}) \subset L_r(\mathbb{R})$. Hint write $f = g + h$ where
   $$g = \begin{cases} f(x) & |f(x)| \geq 1 \\ 0 & \text{otherwise} \end{cases} \qquad h = \begin{cases} f(x) & |f(x)| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

4. If X is a normed linear space and for any A satisfying:
   $$\{x : \|x\| < 1\} \subset A \subset \{x : \|x\| \leq 1\}$$
   and defining
   $$p(x) = \inf\{\lambda > 0 : x \in \lambda A\}$$
   then for $x \in X,\ p(x) = \|x\|$.

5. Let $T : L_2([0,1] \to L_2([0,1]$ be the operator

$$(Tf)(t) = tf(t)$$

. Show T is self-adjoint and compute its norm. Consider the sequence

$$f_n(t) = \begin{cases} \sqrt{n} & 1 - \frac{1}{n} \le x \\ 0 & \text{otherwise} \end{cases}$$

Show $\|f_n\|_2 = 1$ and $\|Tf_n - f_n\|_2 \to 0$. but 1 is not an eigenvalue of T. Some authors would say, 1 is in the approximate spectrum. Why does (f_n) has no convergent subsequence?