# Why Functional Analysis

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## 1 Linear Algebra Envy

Look at all things you get from using linear algebra to solve systems of linear equations of the form

$$Ax = y \tag{1}$$

where $A$ is an $n \times n$ matrix, $x$ and $y$ are column vectors of length $n$.

1. $A$ is a continuous linear function.
2. The system (1) has a solution iff $y$ is in the range of $A$. So there is a solution for all $y$, iff $A$ in onto (surjective).
3. Any solution to (1) is unique iff $\ker A = \{0\}$ iff $A$ is one-one (injective). Any if the solution is not unique, the general solution is $x = x_{\text{particular}} + x_{\text{homogenous}}$
4. If $A$ is one-one onto, the solution $x$ depends continuously on the $y$. Or equivalently, the map $A^{-1}$ is also continuous. (Of course, numerically the inverse could be badly behaved depending on the size of $\det A$ or the norm of $A^{-1}$.)
5. Most interesting operators are symmetric and they have a full set of eigenvectors which correspond to real eigenvalues. In particular, this operators are orthogonally similar to diagonal matrices with real entries.
6. The spectral mapping theorem, the functional calculus and similar tools. For polynomial $p(x)$ the $\textbf{spectrum}(p(A)) = p(\textbf{spectrum}(A))$ and the result generalizes to powers series and analytic functions.

## 2 Continuity requires a Norm

Usually a norm is used to generate the topology for continuity. This is free for finite dimensions in the sense that all 'nice' vector topologies are the same on any finite dimensional space. This follows since bounded closed sets are compact. In infinite dimensions, this is no longer true, there are discontinuous linear functions, so topologies are not unique. The norm also can be used to estimate how bad numerical inverses can be. Sometimes Functional Analysts work in non-normed spaces which requires more notions from general topology.

Sometimes things generalize with more work. The Banach Open Mapping Theorem says a one-one onto continuous linear operator on Banach spaces have continuous inverses.

## 3 But what space to Use?

Consider a differential equations like $Df = g$, the differentiation operator is linear and elementary calculus will tell us that if $g$ is continuous, then the $f = \int g$ is a solution, which is unique up to a constant. What should be our domain for $D$? While one might pick the space $C^1$ of functions with one continuous derivative, but then the range of $D$ is outside this space. While

$$D : C^\infty \to C^\infty$$

has the same space as the domain and range, the topology on $C^\infty$ is more complex than a norm topology. While

$$L : C^1 \to C^0 \oplus \mathbb{R}$$

where

$$L(f) = (f', f(0))$$

is one-one and onto (so the open mapping theorem works), the domain $\neq$ to the range makes eigenfunctions more complex. Finally, it is sometimes an advantage to increase the notation of functions so that more function-like objects have derivatives (like the Dirac delta function).

# 4 Why is Hilbert space a good choice?

The complete spaces with an inner product are called Hilbert spaces and the notion of orthogonally makes all the finite dimensional geometry work but at some cost. Just as completing the rational $\mathbb{Q}$ leads to the irrationals, one completes Riemann integrals and gets Lebesgue integration. While technically Lebesgue is a graduate topic, spectral theory is almost everywhere Riemann integration.