Functional Analysis Problems

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Easy Problems

1. If A is and $n \times n$ matrix with eigenvalue λ and eigenvector x (so that $Ax = \lambda x$) and $p(s)$ is polynomial $c_0 + c_1 s + \cdots + c_k s^k$, then the matrix $p(A)$ has eigenvalue $p(\lambda)$ and eigenvector x. (Spectra Mapping I)

2. If A and B are $n \times n$ matrices both with the same eigenvectors $x_1, \ldots x_n$, but the eigenvalues for A are are the distinct $\lambda_1, \ldots \lambda_n$ and the eigenvalues for B are $\mu_1, \ldots \mu_n$. Show there is a polynomial $p(s)$ so that $p(\lambda_i) = \mu_i$ for $1 \leq i \leq n$. In general, what is the minimal degree of p? Show $p(A) = B$. (Spectra Mapping II)

3. Let A is a 2×2 matrix with eigenvectors x_1, x_2 and eigenvalues $\lambda_1 \neq \lambda_2$. Suppose B is a 2×2 matrix so that $AB = BA$, show x_1, x_2 are also eigenvectors for B and there is a polynomial $p(s)$, so that $p(A) = B$.

4. Let $S : \ell^2 \to \ell^2$ be the *shift* operator given by $S((\alpha_i)_{i\in\mathbb{N}}) = (\beta_i)_{i\in\mathbb{N}}$ where $\beta_1 = 0$ and $\beta_{i+1} = \alpha_i$ for $i \geq 1$. Show S is not onto, but is one to one, in fact show S is an into isometry, which means $\|Sx\| = \|x\|$ for all vectors x.

5. Let $T : \ell^2 \to \ell^2$ be the backwards *shift* operator given by $T((\alpha_i)_{i\in\mathbb{N}}) = (\beta_i)_{i\in\mathbb{N}}$ where $\beta_i = \alpha_{i+1}$ for $i \geq 1$. Show T is not one to one, and it has a one dimensional kernel $\ker T = \{x | Tx = 0\}$ but is onto. Futhermore show TS is the identity, but ST is not.

6. Let $D : \ell^2 \to \ell^2$ be the *diagonal* operator given by $D((\alpha_i)_{i\in\mathbb{N}}) = (\beta_i)_{i\in\mathbb{N}}$ where $\beta_i = \frac{\alpha_i}{i}$ for $i \geq 1$. Show D is not onto, but is one to one.

7. Let $x, y \in H$ a Hilbert space, show $f(t) = \langle x + ty, x + ty\rangle$ is a differentiable function from $\mathbb{R} \to \mathbb{R}$ with a unique minimum at $t = s$ so that $x + sy \perp y$. (This is the point nearest the origin on a line, which is a translate of a one dimensional subspace.)

8. If N is a closed subspace of a Hibert space H and $x \notin N$, there there is a $y \in N^\perp$ so that $\langle y, x\rangle \neq 0$; (so y "separates" x from N).

9. Suppose $\{e_n\}$ is an orthonormal basis for a Hilbert space H. Show no subsequence of $\{e_n\}$ converges (and hence the unit ball $U = \{x \in H | \|x\| \leq 1\}$ is not compact); but for each $x \in H$, $< x, e_n > \to 0$ as $n \to \infty$ (this is called weak converence to 0.)

10. If A is an bounded linear operator on a Hilbert space, that $(\text{image}\, A)^\perp = \ker A^*$ and $(\text{image}\, A^*)^\perp = \ker A$.

11. Show if V is an operator, so that $\|Vx\| = \|x\|$ then $\langle Vx, Vy\rangle = \langle x, y\rangle$, $V^*V = I$ and VV^* is a self adjoint projection. Find S^*, SS^*, and T^* for the shift operators in Problem 4. Define an operator $T_2 \neq T$ that is also a left inverse for S.

12. Let $m(x)$ be a piecewise continuous function. for $0 \leq x \leq 1$ and let A be the operator on $L^2(0, 1)$:

$$(Af)(x) = m(x)f(x)$$

Show $\|A\| = \|m(x)\|_\infty$ is the sup of $|m(x)|$ except for possibility finitely many x. When is A self-adjoint? unitary? a self-adjoint projection?

More challenging

1. Show the parallelogram law characterizes norms that come from inner products. (Hint: do the real case first.)

2. Show ℓ^2 has an uncountable linearly independent subset. (Hint1: find an uncountable collection $\mathscr{C}$ of infinite subsets of $\mathbb{N}$ so that if $A \neq B$, $A, B \in \mathscr{C}$, then $A \cap B$ is a finite set.) (Hint2: There are couple of ways to find $\mathscr{C}$. For each real r find a rational sequence (q_i) so that $q_i \to r$. Or for each θ, find all points $\{(n,m)|n,m \in \mathbb{Z}, |n \sin\theta - m\cos\theta| \leq 1\}$ which are the integer lattice points, (n,m), within one unit of the line through the origin with slope $\tan\theta$.)

3. Suppose H is a non-separable Hilbert space, show there is some uncountable set Γ and an orthonormal set $\{e_\gamma|\gamma \in \Gamma\}$ with dense linear span in H so that $x \in H$, if and only if $\sum |\langle e_\gamma, x\rangle|^2 < \infty$ and $x = \sum \langle e_\gamma, x\rangle e_\gamma$. (Hint: the uncountable sum of positive numbers is easy to define as the sup of all finite partial sums. If the uncountable sum is finite, then at most a countable number are non-zero, and so the sum reduces to the usual sum of a sequence.)

4. More general idempotent (projection like) operators. Let $P^2 = P$ be a bounded linear operator on Hilbert space. Let $X = \{x|Px = x\}$ and $Y = \{x|Px = 0\}$ and let $\Delta = \inf\{\|x - y\| | \|x\| = \|y\| = 1, x \in X, y \in Y\}$ Show $\|P\| = 1/\Delta$.