Reduced Row Echelon Form

Steven Bellenot

May 11, 2008

Reduced Row Echelon Form – A.K.A. rref

For some reason our text fails to define rref (Reduced Row Echelon Form) and so we define it here. Most graphing calculators (TI-83 for example) have a rref function which will transform any matrix into reduced row echelon form using the so called elementary row operations.

We will use Scilab notation on a matrix A for these elementary row operations. In Scilab, row 3 of a matrix A is given by $A(3,:)$ and column 2 is given by $A(:,2)$. The colon acts as a wild card. The elementary row operations are

1. $A(i,:) = A(i,:) + c * A(j,:)$, add a multiple of row j (c times row j) to row i. [We can assume $c \neq 0$.]

2. $A(i,:) = c * A(i,:)$, multiply row i by a non zero constant $c \neq 0$.

3. Swapping two rows, this operation is rarely needed. It requires three Scilab commands:

$$\text{Tmp} = A(i,:); A(i,:) = A(j,:); A(j,:) = \text{Tmp}$$

   [The Tmp, a temporary variable, is needed due to the way variables are stored.]

Scilab will compute the rref, using the rref command, namely

$$\text{rref}(A)$$

.

A matrix is in Reduced Row Echelon Form provided

1. The first non-zero entry in any row is the number 1, these are called **pivots**. (So each row can have zero or one pivot.)

2. A pivot is the only non-zero entry in its column. (So each column can have zero or one pivot.)

3. Rows are orders so that rows of all zeros are at the bottom, and the pivots are in column order.

Examples of matrices that are not in rref

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \text{fails rule 2}$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{fails rule 1}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{fails rule 3}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{fails rule 3}$$

Often a rref matrix is mainly zero's and ones. There can be entries different from zero or one in a rref; here are some examples of rref matrices with other kind of entries.

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 7 \end{bmatrix} \text{no pivot in column 4}$$

$$\begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{no pivot in columns } 2, 4$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{no pivot in column 3}$$

Our textbook uses rref in Gauss-Jordan Elimination. Note that a matrix in rref form is also in the textbook's ref form. Most graphing calculators, for example the TI-83, and Scilab have rref operators which will also do the job. (That is, rref is the name of the operator that does rref.) Most do not have a ref operator

Reduced Row Echelon Form Definition

We give a definition of rref that is similar to the text's ref on page 2. Rule (R3) is replaced by rule (RR3) A matrix is in reduced row echelon form if it satisfies four conditions

(R1): All nonzero rows precede (that is appear above) zero rows when both types are contained in the matrix.

(R2): The first (leftmost) nonzero element of each nonzero row is unity (the number 1).

(RR3): When the first nonzero element of a row appears in column c, then all other elements in column c are zero

(R4): The first nonzero element of any nonzero row appears in a later column (further to the right) than the first nonzero element of any preceding row.

An example

We do row operations on matrix below to convert to rref. (Similar to problem 1.29)

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$$

$$A(2,:) = A(2,:) + 2 * A(1,:) \qquad \text{Multiple row 1 by 2 and add to row 2}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 3 & 5 & 0 \end{bmatrix}$$

$$A(3,:) = A(3,:) - 3 * A(1,:) \qquad \text{Multiple row 1 by -3 and add to row 3}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix}$$

$$A(3,:) = A(3,:) + 1 * A(2,:) \qquad \text{Multiple row 2 by 1 and add to row 3}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

We are now in ref, we continue:

$$A(1,:) = A(1,:) - 2 * A(2,:) \qquad \text{Multiple row 2 by -2 and add to row 1}$$

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix is now in rref.

More often than not

The following collection of commands will often (but not always) put a 3×3 matrix in rref.

$$A(1,:) = A(1,:)/A(1,1) \qquad \text{fails when } A(1,1) = 0$$

$$A(2,:) = A(2,:) - A(2,1) * A(1,:) \qquad \text{assumes } A(1,1) = 1$$
$$A(3,:) = A(3,:) - A(3,1) * A(1,:) \qquad \text{assumes } A(1,1) = 1$$
$$A(2,:) = A(2,:)/A(2,2) \qquad \text{fails when } A(2,2) = 0$$
$$A(1,:) = A(1,:) - A(1,2) * A(2,:) \qquad \text{assumes } A(2,2) = 1$$
$$A(3,:) = A(3,:) - A(3,2) * A(2,:) \qquad \text{assumes } A(2,2) = 1$$
$$A(3,:) = A(3,:)/A(3,3) \qquad \text{fails when } A(3,3) = 0$$
$$A(1,:) = A(1,:) - A(1,3) * A(3,:) \qquad \text{assumes } A(3,3) = 1$$
$$A(2,:) = A(2,:) - A(2,3) * A(3,:) \qquad \text{assumes } A(3,3) = 1$$

If this method works then the resulting matrix is the 3×3 identity matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$