MAP 3306 eMath2 **Test 3** 30 Nov 2007

Directions: Show **ALL** work for credit; Give **EXACT** answers when possible; Start each problem on a **SEPARATE** page; Use only **ONE** side of each page; Be neat; Leave margins on the left and top for the **STAPLE**; Nothing written on this page will be graded;

1. Let $\Delta u = u_{xx} + u_{yy}$, let $c^2 = 1$, let the region S be the square of size $\pi \times \pi$, let ∂S be the boundary of the square region, let the boundary condition $u|_{\partial S} = 0$ be the statement that on all four sides of S, the function u is zero for all time t, and inside this square region let the function f have double Fourier series (yes they are both k's)

$$f(x,y) = \sum_{k=1}^{100} \sin 3kx \sin 4ky$$

   For each problem below is should be straightforward to **IDENTIFY** the type of PDE and to **WRITE** the solution $u(x,y,t)$ (without any λ's).

$$\begin{array}{lllll} A. & \Delta u = u_t; & u|_{\partial S} = 0; & u(x,y,0) = f(x,y) & \\ B. & \Delta u = u_{tt}; & u|_{\partial S} = 0; & u(x,y,0) = f(x,y); & u_t(x,y,0) = 0 \\ C. & \Delta u = u_{tt}; & u|_{\partial S} = 0; & u(x,y,0) = 0; & u_t(x,y,0) = f(x,y) \end{array}$$

2. Use Fourier Transforms to solve $u_x - u_t + u = 0$; $u(x,0) = f(x)$

3. Use the integral definition (3) to find the Fourier Transform of $f(x) = \begin{cases} \cos x & |x| < \pi \\ 0 & \text{otherwise} \end{cases}$

   You might need to use (18) and/or (19) to simplify your answer.

4. True or False and a brief reason why or why not.

   (a) In polar coordinates, the Laplacian is $u_{rr} + u_{\theta\theta}$.

   (b) if $f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$ then the convolution $(g * f)(x) = \int_0^1 g(x-w)\,dw$

   (c) If $A(w)$ is a nice function so that $u(x,t) = \int_0^\infty A(w) \cos wx \exp(-w^2 t)\,dw$ has all its derivatives, then u solves the PDE $u_{xx} = u_t$ for $-\infty < x < \infty$ and $0 < t < \infty$.

   (d) The function $u(x,t) = \frac{1}{\sqrt{4\pi t}} \exp(-\frac{x^2}{4t})$, has Fourier transform $\widehat{u}(w,t)$ is equal to $u(w,t)$.

   (e) The Fourier transform of $\frac{d^3 f(x)}{dx^3}$ is $w^3 \widehat{f}(w)$

   (f) If $u(x,y)$ solves Laplace's equation in the unit square, $u(0,t) = u(1,t) = u(x,0) = 0$, and $u(x,1) = b_2 \sin 2\pi x$ then $u(x, 2^{-1}) = b_2 \sin 2\pi x \sinh(\pi)/\sinh(2\pi)$

   (g) If $u(r,\theta)$ solves Laplace's equation in the unit disk, $u(1,\theta) = a_0 + a_3 \cos 3\theta + b_5 \sin 5\theta$, then $u(3^{-1},\theta) = a_0 + 3^{-3} a_3 \cos n\theta + 5^{-5} b_n \sin 5\theta$,

   (h) If $\xi = -x^2 + y$ and $\eta = x^2 + y$, then $u_{xx} = 2(\eta - \xi)(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})$

   (i) The PDE $x^2 u_{xx} - 2xy u_{xy} + y^2 u_{yy} = (x u_x - y u_y)^2$ is linear.

   (j) If $f(x)$ is odd (real-valued) function with a Fourier transform, then $\widehat{f}$ is purely imaginary and even, that is $\widehat{f}(w) = i g(w)$ where $g(w)$ is both real-valued and even.

The Fourier transform formula's are on the other side.

$$\mathscr{F}[f(x)] = \widehat{f}(w) \text{ or simply } \mathscr{F}[f] = \widehat{f} \tag{1}$$

$$\mathscr{F}^{-1}[\widehat{f}(w)] = f(x) \text{ or simply } \mathscr{F}^{-1}[\widehat{f}] = f \tag{2}$$

$$\mathscr{F}[f(x)](w) = \widehat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx \tag{3}$$

$$\mathscr{F}^{-1}[\widehat{f}(w)](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(w) e^{iwx}\, dw \tag{4}$$

$$\mathscr{F}[u(x,t)](w,t) = \widehat{u}(w,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-iwx}\, dx \tag{5}$$

$$\mathscr{F}^{-1}[\widehat{u}(w,t)](x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{u}(w,t) e^{iwx}\, dw \tag{6}$$

$$\mathscr{F}[af(x) + bg(x)](w) = a\widehat{f}(w) + b\widehat{g}(w) \tag{7}$$

$$\mathscr{F}[f'(x)](w) = iw\widehat{f}(w) \tag{8}$$

$$\mathscr{F}[f''(x)](w) = -w^2\widehat{f}(w) \tag{9}$$

$$\mathscr{F}[\frac{\partial}{\partial x} u(x,t)](w,t) = iw\widehat{u}(w,t) \tag{10}$$

$$\mathscr{F}[\frac{\partial^2}{\partial x^2} u(x,t)](w,t) = -w^2\widehat{u}(w,t) \tag{11}$$

$$\mathscr{F}[\frac{\partial}{\partial t} u(x,t)](w,t) = \frac{\partial}{\partial t}\widehat{u}(w,t) \tag{12}$$

$$\mathscr{F}[\frac{\partial^2}{\partial t^2} u(x,t)](w,t) = \frac{\partial^2}{\partial t^2}\widehat{u}(w,t) \tag{13}$$

$$[f * g](x) = \int_{-\infty}^{\infty} f(w) g(x-w)\, dw = [g * f](x) = \int_{-\infty}^{\infty} f(x-w) g(w)\, dw \tag{14}$$

$$\mathscr{F}[f * g] = \sqrt{2\pi}\,\widehat{f}\widehat{g} \tag{15}$$

$$f(x-a) = \mathscr{F}^{-1}[e^{-iwa}\widehat{f}(w)] \tag{16}$$

$$\mathscr{F}[\exp(-ax^2)] = \frac{1}{\sqrt{2a}} \exp(\frac{-w^2}{4a}) \tag{17}$$

$$\sin wa = \frac{e^{iwa} - e^{-iwa}}{2i} \tag{18}$$

$$\cos wa = \frac{e^{iwa} + e^{-iwa}}{2} \tag{19}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-iwx}\, dx = \sqrt{\frac{2}{\pi}}\, \frac{\sin aw}{w} \tag{20}$$

$$\mathscr{F}\left[\frac{\sin ax}{x}\right] = \sqrt{\frac{\pi}{2}} \text{ if } |w| < a; 0 \text{ otherwise} \tag{21}$$

$$\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ax} e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}(a+iw)} \tag{22}$$