PDE's that can be solved like ODEs

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1 The method in reverse

Perhaps the best way to understand what is going on is to run the process in reverse. We start with an ODE, solve the ODE and show what the coresponding PDEs and PDE solutions look like. So our ODE will have $w = w(t)$ so that $w' = \frac{dw}{dt}$ while our PDE will have $u = u(x, y)$ so that $u_x = \frac{\partial u}{\partial x}$ and $u_y = \frac{\partial u}{\partial y}$. We will write down a table for each equation type, one line for the ODE, a line for partial x version and one line for the partial y version.

Simplest ODE:
$w' = 0 \quad w(t) = C_1$
$u_x = 0 \quad u(x, y) = C_1(y)$
$u_y = 0 \quad u(x, y) = C_1(x)$

Simplest second order ODE
$w'' = 0 \quad w(t) = C_1 t + C_2$
$u_{xx} = 0 \quad u(x, y) = C_1(y)x + C_2(y)$
$u_{yy} = 0 \quad u(x, y) = C_1(x)y + C_2(x)$

Simple first order, homogeneous: solve by separation or integrating factor
$w' = w \quad w(t) = C_1 e^t$
$u_x = u \quad u(x, y) = C_1(y)e^x$
$u_y = u \quad u(x, y) = C_1(x)e^y$

First order, needs integrating factor of e^{t^2}
$w' + 2tw = e^{-t^2} \quad w(t) = te^{-t^2} + C_1 e^{-t^2}$
$u_x + 2xu = e^{-x^2} \quad u(x, y) = xe^{-x^2} + C_1(y)e^{-x^2}$
$u_y + 2yu = e^{-y^2} \quad u(x, y) = ye^{-y^2} + C_1(x)e^{-y^2}$

Second order, homogeneous: solve by characteristic polynomial
$w'' + 25w = 0 \quad w(t) = C_1 \cos 5t + C_2 \sin 5t$
$u_{xx} + 25u = 0 \quad u(x, y) = C_1(y) \cos 5x + C_2(y) \sin 5x$
$u_{yy} + 25u = 0 \quad u(x, y) = C_1(x) \cos 5y + C_2(x) \sin 5y$

Simple second order, non-homogeneous: need undetermined coefficients for particular solution.
$w'' + 25w = 96 \cos t \quad w(t) = C_1 \cos 5t + C_2 \sin 5t + 4 \cos t$
$u_{xx} + 25u = 96 \cos x \quad u(x, y) = C_1(y) \cos 5x + C_2(y) \sin 5x + 4 \cos x$
$u_{yy} + 25u = 96 \cos y \quad u(x, y) = C_1(x) \cos 5y + C_2(x) \sin 5y + 4 \cos y$

2 The mixed case, where $p = u_x$ or $p = u_y$

These are the equations where we first substitute either $p = u_x$ or $p = u_y$ and this this reduces the problem to one above, an extra integration gives the final solution. We write $D_n(x)$ for any function that is an antiderivative to $C_n(x)$

Again the simplest. Since the mixed partials $u_{xy} = u_{yx}$ it is a good thing we get same answer. Remember $u_{xy} = (u_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x}$, that is the notations "grow" in opposite directions.
$w' = 0 \quad w(t) = C_1$
$u_{yx} = p_x = 0 \quad u_y = p(x, y) = C_1(y) \quad u(x, y) = D_1(y) + C_2(x)$
$u_{xy} = p_y = 0 \quad u_x = p(x, y) = C_1(x) \quad u(x, y) = D_1(x) + C_2(y)$

The slightly more complex

$$
\begin{array}{lll}
w' - w = 0 & w(t) = C_1 e^t & \\
u_{yx} - u_y = p_x - p = 0 & u_y = p(x,y) = C_1(y)e^x & u(x,y) = D_1(y)e^x + C_2(x) \\
u_{xy} - u_x = p_y - p = 0 & u_x = p(x,y) = C_1(x)e^y & u(x,y) = D_1(x)e^y + C_2(y)
\end{array}
$$

And finally the ugly third order example

$$
\begin{array}{lll}
w'' + 25w = 0 & w(t) = C_1 \cos 5t + C_2 \sin 5t & \\
u_{yxx} + 25u_y = 0 & u_y(x,y) = C_1(y)\cos 5x + C_2(y)\sin 5x & u(x,y) = D_1(y)\cos 5x + D_2(y)\sin 5x + C_3(x) \\
u_{xyy} + 25u_x = 0 & u_x(x,y) = C_1(x)\cos 5y + C_2(x)\sin 5y & u(x,y) = D_1(x)\cos 5y + D_2(x)\sin 5y + C_3(y)
\end{array}
$$