Integration by Parts, Part 2

Steven Bellenot

September 11, 2007

1 The technique is so nice, let's do it twice

Some integrals can be done but using parts twice and then noticing the "leftover" integral is a scalar multiple of the original integral. This too can be done from our table. First we repeat the next section from the earlier text. Our example integrals are $\int e^{2x} \sin 3x \, dx$ and $\int \cos \pi x \cos 2n\pi x$.

2 Generalized parts

Lets rewrite parts as

$$\int fg' = fg - \int f'g$$

and now apply it twice to

$$\int fg'' = fg' - \int f'g' = fg' - (f'g - \int f''g) = fg' - f'g + \int f''g$$

And third time

$$\int fg''' = fg'' - f'g' + f''g - \int f'''g$$

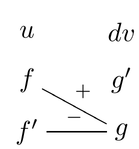
The formulas continue.

$$\int fg'''' = fg''' - f'g'' + f''g' - f'''g + \int f''''g$$

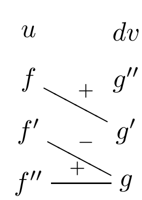
3 Table revisited

We form a tables to express the above formula's. As before, The u column does derivative as we go down while the dv column does anti-derivatives. As before we have the diagonal products, and now the leftover integral is the bottom row's hortizontal product. Both the diagonal's and the hortizontal products alternate signs. Here are the first 3 cases.

$\int fg' = fg - \int f'g$



$\int fg'' = fg' - f'g + \int f''g$



$$\int fg''' = fg'' - f'g' + f''g - \int f'''g$$

u	sign	dv
f	$+$	g'''
f'	$-$	g''
f''	$+$	g'
f'''	$-$	g

4 Example A $\int e^{2x} \sin 3x \, dx$

We form a table to evaluate $I = \int e^{2x} \sin 3x \, dx$

u	dv
e^{2x}	$\sin 3x$
$2e^{2x}$	$\frac{-\cos 3x}{3}$
$4e^{2x}$	$\frac{-\sin 3x}{9}$

Note the bottom row is $-\frac{4}{9}$ times the first.

$$\begin{aligned} I &= -e^{2x}\frac{\cos 3x}{3} + 2e^{2x}\frac{\sin 3x}{9} - \frac{4}{9}I \\ \frac{13}{9}I &= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x \\ I &= -\frac{3}{13}e^{2x}\cos 3x + \frac{2}{13}e^{2x}\sin 3x \end{aligned}$$

We need a constant of integration at the end.

5 Example B $\int \cos \pi x \cos 2n\pi x \, dx$

Again a table to evaluate $I = \int \cos \pi x \cos 2n\pi x \, dx$

u	dv
$\cos \pi x$	$\cos 2n\pi x$
$-\pi \sin \pi x$	$\frac{\sin 2n\pi x}{2n\pi}$
$-\pi^2 \cos \pi x$	$\frac{-\cos 2n\pi x}{(2n\pi)^2}$

Note the bottom row is $\frac{1}{4n^2}$ times the first.

$$\begin{aligned} I &= \cos \pi x\frac{\sin 2n\pi x}{2n\pi} - \pi \sin \pi x\frac{\cos 2n\pi x}{(2n\pi)^2} + \frac{1}{4n^2}I \\ (1 - \frac{1}{4n^2})I &= \frac{1}{2n\pi}\cos \pi x \sin 2n\pi x - \frac{1}{4n^2\pi}\sin \pi x \cos 2n\pi x \\ I &= \frac{1}{(4n^2-1)\pi}\left(2n \cos \pi x \sin 2n\pi x - \sin \pi x \cos 2n\pi x\right) \end{aligned}$$

We need a constant of integration at the end.