Several Views of Integration by Parts

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1 Classic view

The classic integration by parts formula is

$$\int u\,dv = uv - \int v\,du$$

which comes from integrating the product rule $(fg)' = f'g + fg'$. When trying to use this formula on real integrals, there are a number of shortcuts and/or organized techniques that might make actual integration easier. Our example integral is

$$\int x^2 \cos nx\,dx$$

which should illustrate the different methods.

2 Step by step

First we blindly do parts and parts. First $u = x^2$ and hence $dv = \cos nx\,dx$; the next step is to find $du = 2x\,dx$ and $v = \frac{\sin nx}{n}$. The application of parts gives

$$\int x^2 \cos nx\,dx = x^2 \frac{\sin nx}{n} - \int 2x \frac{\sin nx}{n}\,dx$$

and we do the second application with $u = 2x$ and $dv = \frac{\sin nx}{n}\,dx$; again $du = 2\,dx$ and $v = -\frac{\cos nx}{n^2}$. The second application of parts gives

$$\int x^2 \cos nx\,dx = x^2 \frac{\sin nx}{n} + 2x \frac{\cos nx}{n^2} - \int 2 \frac{\cos nx}{n^2}\,dx$$

We can integrate this last intgral to obtain:

$$\int x^2 \cos nx\,dx = x^2 \frac{\sin nx}{n} + 2x \frac{\cos nx}{n^2} - 2 \frac{\sin nx}{n^3} + C$$

(Alternately one can do parts a third time.)

3 Guess and check

This method is basically guess and then correct. First we guess

$$F(x) = x^2 \frac{\sin nx}{n}$$

because the dirivative of the factor containing sin will match the integrand. But the product rule gives another term.

$$F'(x) = x^2 \cos nx + 2x \frac{\sin nx}{n}$$

We need to correct $F(x)$ to cancel this term, guess

$$F(x) = x^2 \frac{\sin nx}{n} + 2x \frac{\cos nx}{n^2}$$

and now

$$F'(x) = x^2 \cos nx + 2x \frac{\sin nx}{n} - 2x \frac{\sin nx}{n} + 2 \frac{\cos nx}{n^2}$$

We have a new term to cancel guess

$$F(x) = x^2 \frac{\sin nx}{n} + 2x \frac{\cos nx}{n^2} - 2 \frac{\sin nx}{n^3}$$

and finally

$$F'(x) = x^2 \cos nx + 2 \frac{\cos nx}{n^2} - 2 \frac{\cos nx}{n^2}$$

as required. Again

$$\int x^2 \cos nx \, dx = x^2 \frac{\sin nx}{n} + 2x \frac{\cos nx}{n^2} + 2 \frac{\sin nx}{n^3} + C$$

Actually all Calculus 2 integration by parts problems are doable by guess and check. See the byguessnbygolly.pdf file.

4 Undetermined coefficients

Here we think of looking for a particular solution to the inhomogeneous problem $y' = x^2 \cos nx$. Undetermined coefficiens say to guess

$$y = Ax^2 \sin nx + Bx^2 \cos nx + Cx \sin nx + Dx \cos nx + E \sin nx + F \cos nx$$

We plug into the ODE

$$y' = Anx^2 \cos nx + 2Ax \sin nx - Bnx^2 \sin nx + 2Bx \cos nx + Cnx \cos nx + C \sin nx - Dnx \sin nx + D \cos nx + En \cos nx - Fn \sin nx$$

Collecting like terms:

$$y' = Anx^2 \cos nx - Bnx^2 \sin nx + (2A - Dn)x \sin nx + (2B + Cn)x \cos nx + (C - Fn) \sin nx + (D + En) \cos nx$$

and we get the system of equations

$$\begin{aligned}
1 &= An \qquad x^2 \cos nx \text{ terms} \\
0 &= -Bn \qquad x^2 \sin nx \text{ terms} \\
0 &= 2B + Cn \qquad x \cos nx \text{ terms} \\
0 &= 2A - Dn \qquad x \sin nx \text{ terms} \\
0 &= D + En \qquad \cos nx \text{ terms} \\
0 &= C - Fn \qquad \sin nx \text{ terms}
\end{aligned}$$

So $A = 1/n$, $B = 0$, $C = 0$, $D = 2/n^2$, $E = -2/n^3$, $F = 0$. and we get the particular solution:

$$y = \frac{1}{n} x^2 \sin nx + \frac{2}{n^2} x \cos nx - \frac{2}{n^3} \sin nx$$

The general solution needs $+C$, which adds the general homogeneous solution. While not needed, a clever person could have figured out that $B = C = F = 0$ at the beginning.

5 Generalized parts

Lets rewrite parts as

$$\int fg' = fg - \int f'g$$

and now apply it twice to

$$\int fg'' = fg' - \int f'g' = fg' - (f'g - \int f''g) = fg' - f'g + \int f''g$$

And third time

$$\int fg''' = fg'' - f'g' + f''g - \int f'''g$$

Applying to $f = x^2$ and $g''' = \cos nx$ gives $f' = 2x$, $f'' = 2$, $f''' = 0$ and $g'' = \sin nx/n$, $g' = -\cos nx/n^2$ and $g = -\sin nx/n^3$ so pluging into the formula

$$\int fg''' = x^2 \sin nx/n + 2x \cos nx/n^2 - 2 \sin nx/n^3$$

The formulas continue.

$$\int fg'''' = fg''' - f'g'' + f''g' - f'''g + \int f''''g$$

6 Polynomial times $g^{(n)}(x)$

We apply the generalized formula's to the special case of a polynomial times a function we can integrate repeatedly. Our polynomial is x^2 our $g^{(n)}(x)$ is $\cos nx$ we form a table. The u column does derivative as we go down while the dv column does anti-derivatives. The third column is a diagonal product with alternating signs. The first entry in the third column comes from the diagonal x^2 to $\sin nx/n$ and ends at $(+1)(x^2)(\sin nx/n)$. The next entry is the diagonal $2x$ to $-\cos nx/n^2$ to $(-1)(2x)(-\cos nx/n^2)$. The last entry is the diagonal 2 to $-\sin nx/n^3$ to $(+1)(2)(-\sin nx/n^3)$. Note the alternating signs (+1), (−1), (+1), etc.

u	dv	
x^2	$\cos nx$	
$2x$	$\sin nx/n$	
2	$-\cos nx/n^2$	$(+1)(x^2)(\sin nx/n)$
0	$-\sin nx/n^3$	$(-1)(2x)(-\cos nx/n^2)$
		$(+1)(2)(-\sin nx/n^3)$

The answer is

$$x^2 \frac{\sin nx}{n} + 2x \frac{\cos nx}{n^2} - 2 \frac{\sin nx}{n^3} + C$$