Laplacian in Other Coordinates Problems and Examples

Steven Bellenot

July 19, 2007

1 Problems

For $u(x,y) = u(\xi,\eta)$ (or $u(x,y,z) = u(\xi,\eta,\zeta)$) express the Laplacian $u_{xx} + u_{yy}$ (or $u_{xx} + u_{yy} + u_{zz}$) in terms of partial derivatives of u with respect to greek varialbes.

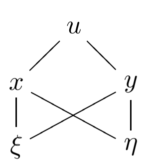
1. $x = \xi + \eta; \quad y = \xi - \eta$
2. $x = 3\xi + 2\eta; \quad y = 2\xi - \eta$ — $ans:\ (5u_{\xi\xi} - 8u_{\xi\eta} + 13u_{\eta\eta})/49$
3. $\xi = \frac{1}{2}(x^2 - y^2); \quad \eta = xy$
4. $\xi = x^2 - y; \quad \eta = x$ — $ans: 2u_\xi + (4\eta^2 + 1)u_{\xi\xi} + 4\eta u_{\xi\eta} + u_{\eta\eta}$
5. $x = \xi + \eta + \zeta; \quad y = \xi - \eta + \zeta; \quad z = \xi - \eta - \zeta$
6. Show that for spherical coordinates (ρ,θ,ϕ) with $x = \rho\cos\theta\sin\phi; \quad y = \rho\sin\theta\sin\phi; \quad z = \rho\cos\phi$ that

$$u_{\rho\rho} + \frac{2}{\rho}u_\rho + \frac{1}{\rho^2}\left(u_{\phi\phi} + \cot\phi\, u_\phi + \frac{1}{\sin^2\phi}\, u_{\theta\theta}\right) = u_{xx} + u_{yy} + u_{zz}$$

2 Examples

- Problem #2 There are at least two ways to do these problems. All require use of the Chain rule for several variables.

  1. The first method takes the variables as they come. Here they are $x = 3\xi + 2\eta; \quad y = 2\xi - \eta$



We crank out the partials $u_\xi, u_\eta, u_{\xi\xi}, u_{\xi\eta}, u_{\eta\eta}$

$$\begin{aligned}
u_\xi &= u_x x_\xi + u_y y_\xi = 3u_x + 2u_y \\
u_\eta &= u_x x_\eta + u_y y_\eta = 2u_x - u_y \\
u_{\xi\xi} &= 3(u_{xx}x_\xi + u_{xy}y_\xi) + 2(u_{yx}x_\xi + u_{yy}y_\xi) = 9u_{xx} + 12u_{xy} + 4u_{yy} \\
u_{\xi\eta} &= 3(u_{xx}x_\eta + u_{xy}y_\eta) + 2(u_{yx}x_\eta + u_{yy}y_\eta) = 6u_{xx} + u_{xy} - 2u_{yy} \\
u_{\eta\eta} &= 2(u_{xx}x_\eta + u_{xy}y_\eta) - (u_{yx}x_\eta - u_{yy}y_\eta) = 4u_{xx} - 4u_{xy} + u_{yy}
\end{aligned}$$

We want combinations of the greek partials that add up to $u_{xx} + u_{yy}$ and often one can find the answer by inspection. But here the problem is harder. We need to solve $Au_{\xi\xi} + Bu_{\xi\eta} + Cu_{\eta\eta} =$

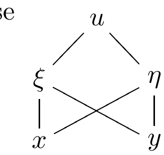
$u_{xx} + u_{yy}$ which gives 3 equations

$$\begin{aligned}9A + 6B + 4C &= 1\\ 12A + B - 4C &= 0\\ 4A - 2B + C &= 1\end{aligned}$$

in three unknowns. Eventually $A = \frac{5}{49}$, $B = -\frac{8}{49}$, and $C = \frac{13}{49}$ giving the answer

$$\frac{1}{49}\left(5u_{\xi\xi} - 8u_{\xi\eta} + 13u_{\eta\eta}\right)$$

2. The second method starts by inverting the equations so we can use



Certainly this is straightforward for this problem as it is a linear equation we get $\xi = \frac{1}{7}(x + 2y)$ and $\eta = \frac{1}{7}(2x - 3y)$. Now we can directly compute $u_{xx} + u_{yy}$ using the chain rule.

$$\begin{aligned}u_x &= u_\xi \xi_x + u_\eta \eta_x = \frac{1}{7}u_\xi + \frac{2}{7}u_\eta\\ u_y &= u_\xi \xi_y + u_\eta \eta_y = \frac{2}{7}u_\xi - \frac{3}{7}u_\eta\\ u_{xx} &= \frac{1}{7}(u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x) + \frac{2}{7}(u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x) = \frac{1}{49}(u_{\xi\xi} + 4u_{\xi\eta} + 4u_{\eta\eta})\\ u_{yy} &= \frac{3}{7}(u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y) - \frac{3}{7}u_{\eta\xi}\xi_y - u_{\eta\eta}\eta_y) = \frac{1}{49}(4u_{\xi\xi} - 12u_{\xi\eta} + 9u_{\eta\eta})\end{aligned}$$

The next step is direct: we add $u_{xx} + u_{yy}$ and get the same answer.

$$\frac{1}{49}\left(5u_{\xi\xi} - 8u_{\xi\eta} + 13u_{\eta\eta}\right)$$

The second method is faster this time.

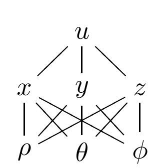
- Problem #4 This one is already in the form of method 2 above. So we take $\xi = x^2 - y; \quad \eta = x$ and directly compute

$$\begin{aligned}u_x &= u_\xi \xi_x + u_\eta \eta_x = 2xu_\xi + u_\eta\\ u_y &= u_\xi \xi_y + u_\eta \eta_y = -u_\xi\\ u_{xx} &= 2u_\xi + 2x(u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x) + (u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x) = 2u_\xi + 4x^2u_{\xi\xi} + 4xu_{\xi\eta} + u_{\eta\eta}\\ u_{yy} &= -(u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y)) = u_{\xi\xi}\end{aligned}$$

and thus $u_{xx} + u_{yy} = 2u_\xi + (4x^2 + 1)u_{\xi\xi} + 4xu_{\xi\eta} + u_{\eta\eta}$ This expression is not free of x and y, but this is easily fixed as $y = -\xi + \eta^2$ and $x = \eta$. Final answer

$$u_{xx} + u_{yy} = 2u_\xi + (4\eta^2 + 1)u_{\xi\xi} + 4\eta u_{\xi\eta} + u_{\eta\eta}$$

- Problem #6 We use $x = \rho\cos\theta\sin\phi; \quad y = \rho\sin\theta\sin\phi; \quad z = \rho\cos\phi$ so that



$$
\begin{aligned}
u_\rho &= u_x x_\rho + u_y y_\rho + u_z z_\rho \\
u_\rho &= u_x \cos\theta \sin\phi + u_y \sin\theta \sin\phi + u_z \cos\phi \\
u_\theta &= u_x x_\theta + u_y y_\theta + u_z z_\theta \\
u_\theta &= -\rho u_x \sin\theta \sin\phi + \rho u_y \cos\theta \sin\phi + 0 \\
u_\phi &= u_x x_\phi + u_y y_\phi + u_z z_\phi \\
u_\phi &= \rho u_x \cos\theta \cos\phi + \rho u_y \sin\theta \cos\phi - \rho u_z \sin\phi \\
u_{\rho\rho} &= u_{xx} \cos^2\theta \sin^2\phi + u_{xy} \cos\theta \sin\theta \sin^2\phi + u_{xz} \cos\theta \sin\phi \cos\phi \\
&\quad + u_{yx} \cos\theta \sin\theta \sin^2\phi + u_{yy} \sin^2\theta \sin^2\phi + u_{yz} \sin\theta \sin\phi \cos\phi \\
&\quad + u_{zx} \cos\theta \sin\phi \cos\phi + u_{zy} \sin\theta \sin\phi \cos\phi + u_{zz} \cos^2\phi \\
u_{\rho\rho} &= u_{xx} \cos^2\theta \sin^2\phi + u_{yy} \sin^2\theta \sin^2\phi + u_{zz} \cos^2\phi \\
&\quad + 2u_{xy} \cos\theta \sin\theta \sin^2\phi + 2u_{xz} \cos\theta \sin\phi \cos\phi + 2u_{yz} \sin\theta \sin\phi \cos\phi \\
u_{\theta\theta} &= -\rho u_x \cos\theta \sin\phi - \rho u_y \sin\theta \sin\phi \\
&\quad + \rho^2 u_{xx} \sin^2\theta \sin^2\phi + \rho^2 u_{yy} \cos^2\theta \sin^2\phi - 2\rho^2 u_{xy} \cos\theta \sin\theta \sin^2\phi \\
u_{\phi\phi} &= -\rho u_x \cos\theta \sin\phi - \rho u_y \sin\theta \sin\phi - \rho u_z \cos\phi \\
&\quad + \rho^2 u_{xx} \cos^2\theta \cos^2\phi + \rho^2 u_{yy} \sin^2\theta \cos^2\phi + \rho^2 u_{zz} \sin^2\phi \\
&\quad + 2\rho^2 u_{xy} \cos\theta \sin\theta \cos^2\phi - 2\rho^2 u_{xz} \cos\theta \sin\phi \cos\phi - 2\rho^2 u_{yz} \sin\theta \sin\phi \cos\phi
\end{aligned}
$$

Lets start by looking at the second order terms, namely $u_{\rho\rho} + u_{\theta\theta}/(\rho^2 \sin^2\phi) + u_{\phi\phi}/\rho^2$, by checking the coefficients of $u_{xx}, u_{xy}, \ldots$

$$
\begin{aligned}
u_{xx} &: \cos^2\theta \sin^2\phi + \sin^2\theta + \cos^2\theta \cos^2\phi = \cos^2\theta(\sin^2\phi + cos^2\phi) + \sin^2\theta = 1 \\
u_{yy} &: \sin^2\theta \sin^2\phi + \cos^2\theta + \sin^2\theta \cos^2\phi = \sin^2\theta(\sin^2\phi + cos^2\phi) + \cos^2\theta = 1 \\
u_{zz} &: \cos^2\phi + 0 + \sin^2\phi = 1 \\
u_{xy} &: 2\cos\theta \sin\theta \sin^2\phi - 2\cos\theta \sin\theta + 2\cos\theta \sin\theta \cos^2\phi = 2\cos\theta \sin\theta(\sin^2\phi + \cos^2\phi) - 2\cos\theta \sin\theta = 0 \\
u_{xz} &: 2\cos\theta \sin\phi \cos\phi + 0 - 2\cos\theta \sin\phi \cos\phi = 0 \\
u_{yz} &: 2\sin\theta \sin\phi \cos\phi + 0 - 2\sin\theta \sin\phi \cos\phi = 0
\end{aligned}
$$

So only the product rule terms are left over. They are

$$
\begin{aligned}
\frac{u_{\theta\theta}}{\rho^2 \sin^2\phi} &: \frac{-u_x \cos\theta - u_y \sin\theta}{\rho \sin\phi} = \frac{-u_x \cos\theta(\cos^2\phi + \sin^2\phi) - u_y \sin\theta(\cos^2\phi + \sin^2\phi)}{\rho \sin\phi} \\
\frac{u_{\phi\phi}}{\rho^2} &: \frac{-u_x \cos\theta \sin\phi - u_y \sin\theta \sin\phi - u_z \cos\phi}{\rho} = \frac{-u_x \cos\theta \sin^2\phi - u_y \sin\theta \sin^2\phi - u_z \cos\phi \sin\phi}{\rho \sin\phi} \\
\text{left over} &= \frac{-2u_x \cos\theta \sin^2\phi - 2u_y \sin\theta \sin^2\phi - u_z \cos\phi \sin\phi - u_x \cos\theta \cos^2\phi - u_x \sin\theta \cos^2\phi}{\rho \sin\phi}
\end{aligned}
$$

Lets checks these versus the first order terms

$$
\begin{aligned}
\frac{2}{\rho} u_\rho &= \frac{2(u_x \cos\theta \sin\phi + u_y \sin\theta \sin\phi + u_z \cos\phi)}{\rho} = \frac{2u_x \cos\theta \sin^2\phi + 2u_y \sin\theta \sin^2\phi + 2u_z \cos\phi \sin\phi}{\rho \sin\phi} \\
\frac{u_\phi \cot\phi}{\rho^2} &= \frac{\cos\phi(u_x \cos\theta \cos\phi + u_y \sin\theta \cos\phi - u_z \sin\phi)}{\rho \sin\phi} = \frac{u_x \cos\theta \cos^2\phi + u_y \sin\theta \cos^2\phi - u_z \cos\phi \sin\phi}{\rho \sin\phi} \\
\text{first order terms} &= \frac{2u_x \cos\theta \sin^2\phi + 2u_y \sin\theta \sin^2\phi + u_z \cos\phi \sin\phi + u_x \cos\theta \cos^2\phi + u_y \sin\theta \cos^2\phi}{\rho \sin\phi}
\end{aligned}
$$

and everything matches.