By Guess and By Golly Integration

Since we skipped Sections 7.3 and 7.4 several times we have had to integrate integrals like

$$\int xe^x\ dt.$$

This short note gives the fastest and easiest way to integrate many integrals like this. Not only that but often the steps can be done via the TI-89 without anyone being the wiser. This method only works on a few integrals, the most common being like the integral above or similar ones like

$$\int x\sin x\ dx, \qquad \int x^2e^{2x}\ dx, \qquad \int \ln x\ dx, \qquad \int \arctan x\ dx$$

Let's consider $\int xe^x\ dx$ and consider the e^x portion. Now if $y = f(x)e^x$, then the product rule is going to give $y' = f(x)e^x + f'(x)e^x$. That is a reasonable guess for $\int f(x)e^x\ dx$ is $f(x)e^x$, but the guess is wrong by $f'(x)e^x$. So we guess $y = xe^x$ for the integral but $y' = xe^x + e^x$ so we have an error of $-e^x$. So to get rid of the error, we must add $g(x)$ to xe^x so that $g'(x)$ is $-e^x$. But this is just $g(x) = -e^x$. So $xe^x - e^x$ is the integral. We can summarized using a table.

Goal	Guess	Result	Error
xe^x	xe^x	$xe^x + e^x$	$-e^x$
$-e^x$	$-e^x$	$-e^x$	0
xe^x	$xe^x - e^x$	xe^x	0

Lets repeat for $\int x^2\sin x\ dx$. This time we start with $-f(x)\cos x$ because the derivative of $-\cos x$ is $\sin x$. We go directly to the table.

Goal	Guess	Result	Error
$x^2\sin x$	$-x^2\cos x$	$x^2\sin x + 2x\cos x$	$-2x\cos x$
$-2x\cos x$	$-2x\sin x$	$-2x\cos x - 2\sin x$	$2\sin x$
$2\sin x$	$-2\cos x$	$2\sin x$	0
$x^2\sin x$	$-x^2\cos x - 2x\sin x - 2\cos x$	$x^2\sin x$	0

Now for $\int 10xe^{-3x}\ dx$, again straight to the tables. But note that the derivative of $f(x)e^{-3x}$ adds a (-3) factor which means we have to divide our guesses by $(-1/3)$.

Goal	Guess	Result	Error
$10xe^{-3x}$	$(-10/3)xe^{-3x}$	$10xe^{-3x} + (-10/3)e^{-3x}$	$(10/3)e^{-3x}$
$(10/3)e^{-3x}$	$(-10/9)e^{-3x}$	$(10/3)e^{-3x}$	0
$10xe^{-3x}$	$(-10/3)xe^{-3x} + (10/9)e^{-3x}$	$10xe^{-3x}$	0

The second kind of functions where this work are for ugly functions with rational or polynomial like derivatives. Note that $\arctan x$, $\arcsin x$ and $\ln x$ have these kind of derivatives. The first step is to observe that the derivative of $xf(x)$ is $f(x) + xf'(x)$. Lets do $\int \ln x\ dx$.

Goal	Guess	Result	Error
$\ln x$	$x\ln x$	$\ln x + x/x$	-1
-1	x	-1	0
$\ln x$	$x\ln x - x$	$\ln x$	0

Finally lets do $\int \arctan x\ dx$ —- here is the table.

Goal	Guess	Result	Error
$\arctan x$	$x\arctan x$	$\arctan x + x/(1+x^2)$	$-x/(1+x^2)$
$-x/(1+x^2)$	$(-1/2)\ln\|1+x^2\|$	$-x/(1+x^2)$	0
$\arctan x$	$x\arctan x - (1/2)\ln(1+x^2)$	$\arctan x$	0