GEAR LECTURES ON QUANTUM HYPERBOLIC GEOMETRY

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1. Introduction

The introduction is the hardest to read part of these notes. Maybe skip it for now, and read it at the end to figure out what happened. You should think of these lectures as a multilayered experience. In the actual lectures I will cover much less ground than in the notes, focusing on the main points. Lectures Ia and Ib are a simplified version of arXiv:1707.09234 , *Unicity for Representations of the Kauffman Bracket Skein Algebra* by Frohman, Kania-bartoszynska and Lê. Lectures IIa, and IIb are a simplified introduction to the work of Bonahon, Wong and Liu.

Quantum hyperbolic geometry is a term coined by Baseilhac and Benedetti,[6] to refer to a method for assigning invariants to a 3-manifold equipped with a representation of its fundamental group into $SL_2\mathbb{C}$. There is a system of $6j$-symbols, parametrized by a complex variable, associated to representations of the cyclic Weil algebra. They realized that the variable acted as a root of the crossratio of an ideal tetrahedron. Basing their work on Kashaev's original approach to defining his invariant they were able to assign invariants to a three manifold equipped with an ideal triangulation that was decorated with numbers that satisfied equations similar to Thurston's consistency conditions for a hyperbolic structure carried by an ideal triangulation.

Thurston's proof of the hyperbolization theorem for sufficiently large acylindrical three-manifolds relied on finding fixed points for the action of the mapping class group on character varieties of surface groups. Bonahon's approach to quantum hyperbolic geometry [8, 9, 10, 11] is via an analogy with Thurston's work. Recent work of Baseilhac and Benedetti, [7], shows that their invariants and Bonahon's are the same.

The goal of these lectures is to introduce the concepts that underly Bonahon's approach to quantum hyperbolic invariants of surface bundles over a circle and the means to compute them.

The Kauffman bracket skein algebra $K_\zeta(F)$ of an oriented finite type surface at a $2n$th root of unity $\zeta \in \mathbb{C}$, where n is odd, is a finite rank noncommutative ring extension of the coordinate ring of the $SL_2\mathbb{C}$ -character variety of the fundamental group of F. The failure of the algebra to be commutative is a reflection of the symplectic geometry of the character variety. The action of the mapping class group of F on the coordinate ring of the $SL_2\mathbb{C}$-character variety extends to $K_\zeta(F)$. This action encodes how the mapping class interacts with the geometry of the character variety.

An irreducible representation of $K_\zeta(F)$ is an onto algebra homomorphism

$$h : K_\zeta(F) \to M_N(\mathbb{C}) \tag{1}$$

where $M_N(\mathbb{C})$ is the ring of $N \times N$-matrices with complex coefficients for some N. Generically, there is a one to one correspondence between irreducible representations of $K_\zeta(F)$ and points of the $SL_2\mathbb{C}$-character variety of F. This means that a fixed point of a mapping class of F on the character variety gives rise to a unique automorphism of a matrix algebra. Quantum hyperbolic invariants of three-manifolds are derived from this automorphism.

Skein algebras have complicated defining relations. There is another class of algebras, **noncommutative tori** that have very simple defining relations. Bonahon and Wong defined an injective homomorphism from the Kauffman bracket skein algebra of a punctured surface to a noncommutative torus. Irreducible representations of the skein algebra can be constructed by pulling back irreducible representations of the noncommutative torus to the skein algebra via this homomorphism. This leads to the ability to compute quantum hyperbolic invariants.

The lectures will be in four parts.

- **Lecture Ia** The Kauffman bracket skein algebra $K_\zeta(F)$ of an oriented surface of finite type F, at a $2n$th root of unity ζ where n is odd, is a prime, affine algebra that has finite rank over its center. Its center is a finite extension of the coordinate ring of the $SL_2\mathbb{C}$-character variety of the fundamental group of F. Furthermore there is a natural action of the mapping class group of F, on $K_\zeta(F)$ as automorphisms. In this lecture we define the Kauffman bracket skein algebra, explain its connection to the coordinate ring of the character variety, and outline its algebraic properties.

- **Lecture Ib** The second part of the first lecture is about the representation theory of associative algebras. If A is an associative algebra over $\mathbb{C}$, then a representation is just a surjective homomorphism from A to the algebra of $n \times n$-matrices with complex coefficients. We will begin by showing that up to equivalence, representations are determined by their kernel. If A is affine (a quotient of an algebra of noncommuting polynomials in finite many variables), has finite rank over its center, and is prime (the noncommutative analog of being an integral domain), then generically irreducible representations are classified by their restriction to the center of the algebra, and generically they all have the same dimension. This is the unicity theorem.

Generically, a fixed point of the action of the mapping class $\phi : F \to F$ on the character variety of the fundamental group of F, and a choice of an odd integer n, gives rise to a unique automorphism of a matrix algebra. This automorphism carries the "quantum hyperobolic" invariants " of the mapping cylinder of ϕ.

- **Lecture IIa** In this lecture we introduce another class of algebras that satisfy the hypotheses of the unicity theorem, noncommutative tori. We describe the embedding of the skein algebra of the torus into a noncommutative torus due to Frohman and Gelca. We show that the rank of these two algebras over their centers are the same, and a local basis for the skein algebra of the torus gets sent to a local basis for noncommutative torus. We finish by discussing the noncommutative A-polynomial.
- **Lecture IIb** We begin by giving standard models for the irreducible representations of the noncommutative torus. Next we describe an action of $SL_2\mathbb{Z}$ on the noncommutative torus so that embedding of the skein algebra into the noncommutative torus is an intertwiner for the two actions. We finish by computing the quantum hyperbolic invariants of the mapping class $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

There is much more material in these written notes than I could possibly explain in four fortyfive minute lectures. You should think of these notes as a companion that fills out the points I make at the board.

2. Lecture Ia: The Kauffman bracket skein algebra

2.1. **Kauffman Bracket Skein Module.** Let M be an oriented 3-manifold. A **framed link** in M is an embedding of a disjoint union of annuli into M. Diagrammatically we depict framed links by showing the core of the annuli. You should imagine the annuli lying parallel to the plane of the paper, this is sometimes called **the blackboard framing**. Two framed links in M are equivalent if they are isotopic. Let $\mathcal{L}$ denote the set of equivalence classes of framed links in M, including the empty link.

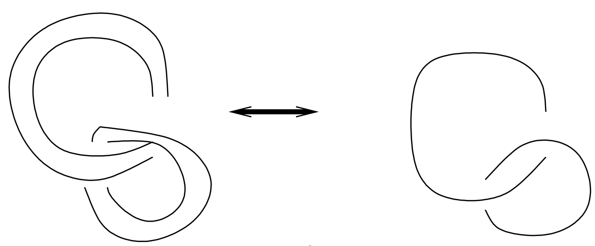
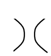


Figure 1. Representing a framed link with the blackboard framing

Let $A \in \mathbb{C}$ be nonzero. Consider the vector space $\mathbb{C}\mathcal{L}$, with basis $\mathcal{L}$. Let S be the subvector space spanned by the Kauffman bracket skein relations,

 $- A$  $- A^{-1}$ 

and

$$\bigcirc \cup L + (A^2 + A^{-2})L.$$

The framed links in each expression are identical outside the balls pictured in the diagrams, and when both arcs in a diagram lie in the same component of the framed link, the same side of the annulus is up. The problem is you could put a crossing ball in a manifold in such a way that one of the smoothings is a pair of Mobius bands. Never apply a skein relation in a way that does not produce annuli. The Kauffman bracket skein module $K_A(M)$ is the quotient

$$\mathbb{C}\mathcal{L}/S(M).$$

A **skein** is an element of $K_A(M)$.

Skeins are equivalence classes of linear combinations of isotopy classes of framed links.

Let F be a compact orientable surface and let $I = [0,1]$. There is an algebra structure on $K_A(F \times I)$ that comes from laying one framed link over the other. The resulting algebra is denoted $K_A(F)$ to emphasize that it comes from the particular structure as a cylinder over F. Denote the stacking product with a $*$, so $\alpha * \beta$ means α stacked over β. If it is known the two skeins commute the $*$ will be omitted.

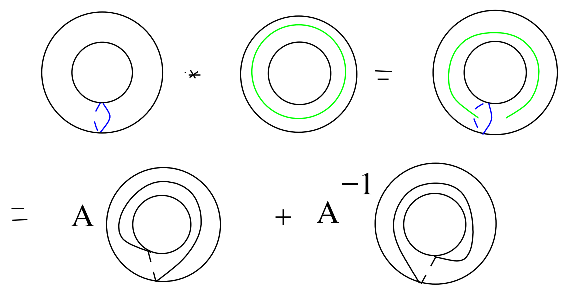


FIGURE 2. The product of two skeins in a cylinder over a torus. In the first row we lay one over the other. In the second row we resolve the crossing

If $\phi : F \to F$ is an orientation preserving homeomorphism, then it extends to an orientation preserving homeomorphism of $\tilde{\phi} : F \times [0,1] \to F \times [0,1]$ by

$$\tilde{\phi}(x,t) = (\phi(x), t). \tag{2}$$

The mapping $\tilde{\phi}$ takes framed links to framed links, and because it doesn't change the last coordinate, gives rise to an automorphism of $K_A(F)$. Hence the mapping class group of a surface acts as automorphisms of the Kauffman bracket skein algebra of the surface.

A **simple diagram** D on the surface F is a system of disjoint simple closed curves so that none of the curves bounds a disk. A simple diagram D is **primitive** if no two curves in the diagram cobound an annulus. A simple diagram can be made into a framed link by choosing a system of disjoint annuli in F so that each annulus has a single curve in the diagram as its core. That is we assume the blackboard framing. The set of isotopy classes of blackboard framed simple diagrams form a basis for $K_A(F)$. Hence every skein in a cylinder over F can be written uniquely as a linear combination of isotopy classes of simple diagrams.

We are most interested in the case when A is a primitive $2n$th root of unity, where $n \in \mathbb{N}$ is odd. To emphasize that we are working at a root of unity we will denote the variable in the Kauffman bracket skein relation by ζ when it is a root of unity.

2.2. **Skeins and characters.** Assume

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3}$$

where $ad - bc = 1$. Notice that $Tr(A) = a + d$. The characteristic polynomial of A is

$$det(A - \lambda Id_2) = \lambda^2 - (a+d)\lambda + 1 = \lambda^2 - Tr(A)\lambda + 1 \tag{4}$$

By the Cayley-Hamilton identity,

$$A^2 - Tr(A)A + Id_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5}$$

Let's multiply through by A^{-1} to get rid of the square.

$$A - Tr(A)Id_2 + A^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{6}$$

Multiplying by an arbitrary matrix B, and taking the trace, which is a linear function we get,

$$Tr(AB) - Tr(A)Tr(B) - Tr(A^{-1}B) = 0. \tag{7}$$

This is the fundamental trace identity for $SL_2\mathbb{C}$. The derivation I gave here is an example of polarization, which is a process for turning a polynomial in a single variable of degree n, into a multilinear function in n variables. For that reason, it is also called **the fully polarized Cayley-Hamilton Identity.**

Recall

- If $A, B \in SL_2\mathbb{C}$, $Tr(AB) = Tr(BA)$. This implies

$$Tr(ABA^{-1}) = Tr(B). \tag{8}$$

- If $A \in SL_2\mathbb{C}$ then $Tr(A) = Tr(A^{-1})$, and
- $Tr(Id_2) = 2$.

Let Σ be a surface of genus 1 with one boundary component. We show Σ in Figure 2.2. Choose two oriented crosscuts, (shown in blue and red), that cut Σ open to be a disk, and choose a basepoint disjoint from the crosscuts. Given a based loop, (shown in green) perturb it to be transverse to the crosscuts. Every time you travserse the blue crosscut write down the letter a if you are

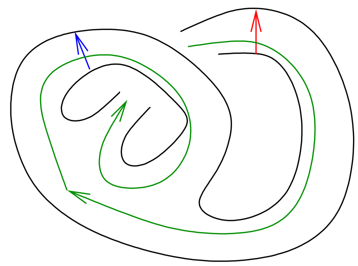


FIGURE 3. The surface Σ.

traversing it in the positive direction and a^{-1} if you travserse it in the negative direction. Here positive means that the local intersection number of the loop with the arc is positive. Similarly, every time you traverse the red crosscut write down b or b^{-1}. Each homotopy class of based loops on the surface corresponds to a freely reduced word in $a^{\pm 1}$ and $b^{\pm 1}$. This correspondence exhibits the fundamental group Σ as the free group $F < a, b >$ on two generators a and b.

For every choice of matrices $(A, B) \in SL_2\mathbb{C}$, there is a homomorphism $\rho : \pi_1(\Sigma) \to SL_2\mathbb{C}$ obtained by substituting $A^{\pm 1}$ and $B^{\pm 1}$ for $a^{\pm 1}$ and $b^{\pm 1}$ in elements of the free group on a and b .

Let G be a finitely generated group. Let $R(G)$ be the set of representations of the fundamental group of Σ into $SL_2\mathbb{C}$. The set $R(G)$ can be given the structure of an algebraic set. If $x_1, \ldots, x_n$ are the generators of G, then each element $\rho : G \to SL_2\mathbb{C}$ is determined by the tuple $(\rho(x_1), \ldots, \rho(x_n)) \in SL_2\mathbb{C}^n$. There is a finite system of equations that characterize the image of $R(G)$ in $SL_2\mathbb{C}$ that are derived from the relations in the group.

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{C}$, you can think of it as $(a, b, c, d) \in \mathbb{C}^4$, with $ad - bc = 1$. Hence the coordinate ring of $SL_2\mathbb{C}$ is the quotient of polynomials in a, b, c, d modulo the ideal generated by $ad - bc - 1$. The coordinate ring $C[SL_2\mathbb{C}^n]$ of $SL_2\mathbb{C}^n$ can be thought of as the tensor product of n copies of this ring. Finally, the coordinate ring of the representation variety $C[R(G)]$ is the quotient of $C[SL_2\mathbb{C}^n]$, coming from saying two functions are equal if they evaluate as the same functions on the set $R(G)$.

There is an action of $SL_2\mathbb{C}$ on $R(G)$ given by conjugation. If $\rho \in R(G)$ and $A \in SL_2\mathbb{C}$, define $A.\rho$ as follows. If $g \in G$ then

$$A.\rho(g) = A\rho(g)A^{-1}. \tag{9}$$

We say that two $\rho_1, \rho_2 : G \to SL_2\mathbb{C}$ are conjugate if there exists $A \in SL_2\mathbb{C}$ so that $A.\rho_1 = \rho_2$. The quotient space under this action is not Hausdorff, which means you can't detect points from continuous coordinate functions that take on values in $\mathbb{C}$. There is a coarser notion of equivalence of representations that leads to a Hausdorff space. We say that $\rho_1, \rho_2 : G \to SL_2\mathbb{C}$ are **trace equivalent** if for every $g \in G$,

$$Tr(\rho_1(g)) = Tr(\rho_2(g)). \tag{10}$$

Since conjugate matrices have the same trace, if ρ_1 and ρ_2 are conjugate representations, then they are trace equivalent. However, by default, you can detect when two representations are trace equivalent, by looking at traces. The set of trace equivalent classes of representations is called the $SL_2\mathbb{C}$-character variety of G.

The left action of $SL_2\mathbb{C}$ on $R(G)$ gives rise to a right action of $SL_2\mathbb{C}$ on $C[R(G)]$. If $f : R(G) \to \mathbb{C}$, $A \in SL_2\mathbb{C}$, and $\rho : G \to SL_2\mathbb{C}$ is a representation then

$$f(\rho).A = f(A.\rho). \tag{11}$$

The $SL_2\mathbb{C}$-character ring $\mathcal{X}(G)$ is the subring of $C[R(G)]$ that is fixed under this action. There is a one-to-one correspondence between trace equivalence classes of representations of G into $SL_2\mathbb{C}$ and maximal ideals of the ring $\mathcal{X}(G)$.

A fancy way to say this is that the character variety is the **categorical quotient** of the representation variety. The category, is the category of algebraic sets and algebraic mappings. The character variety is an algebraic set, which means that it corresponds in a one-to-one fashion with the zeroes of a collection of polynomials.

The description of the ring of characters is unambiguous, but hard to use. **Classical invariant theory** was aimed at finding descriptions of character rings that are easy to compute with. The first fundamental theorem of classical invariant theory supplies a spanning set of the ring of characters, and the second fundamental theorem of classical invariant theory gives all relations between them.

Start by assuming F is the free group on $x_1, \dots x_n$. The conjugacy classes of F are in one to one correspondence with freely reduced cyclic words in $x_i^{\pm 1}$. We consider two words to be equal if one is the cyclic permutation of the other, and we only consider such words so that no generator and its inverse appear next to one another in any cyclic rotant of the word. Hence $x_1x_2x_1^{-1}$ is not freely cyclically reduced because we can cyclically rotate it to get $x_1^{-1}x_1x_2$ which is not freely reduced. The words x_1x_2 and x_2x_1 are cyclically equivalent. Let $S(F)$ be the polynomial algebra where the variables are cyclic equivalence classes of freely cyclically reduced words. There is an algebra homomorphism

$$\Theta : S(F) \to \mathcal{X}(F) \tag{12}$$

sending the equivalence class of the freely cyclically reduced word X to the function that sends the representation $\rho : F \to SL_2\mathbb{C}$ to

$$-Tr(\rho(X)). \tag{13}$$

The first fundamental theorem of classical invariant theory says that this map is onto. The second fundamental theorem of classical invariant theory says that the kernel of Θ is generated as an ideal by

- The polynomial that says the trace of the identity is 2, that is

$$(e) + 2. \tag{14}$$

- Polynomials that say that the trace of a matrix is equal to its inverse, that is for all equivalence classes of freely cyclically reduced words,

$$(X) - (X^{-1}) \tag{15}$$

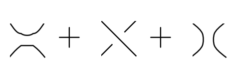
- Finally, functional evaluation of the fully polarized Cayley-Hamilton Identity. If X and Y are freely cyclically reduced words,

$$(X)(Y) + (XY) + (XY^{-1}). \tag{16}$$

This is an adaptation of the work of Procesi, [22] to $SL_2\mathbb{C}$ by Bullock, [14].

If G is a quotient of the free group F , then $\mathcal{X}(G)$ is the quotient of $\mathcal{X}(F)$ by the smallest radical ideal containing all relations between characters that are induced by relations in the group G.

The Kauffman bracket skein relation at $A = -1$ is:

(17)  $= 0.$

Letting $\eta(A) = -Tr(A)$, the trace identity becomes,

$$\eta(A)\eta(B) + \eta(AB) + \eta(A^{-1}B) = 0. \tag{18}$$

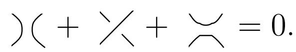
The other Kauffman bracket skein relation at $A = -1$ is,

$$\bigcirc \cup L + 2L.$$

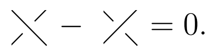
and since $\eta(Id_2) = -2$, we have

$$\eta(Id_2)\eta(A) + 2\eta(A) = 0. \tag{19}$$

This would lead you to believe there is a connection between $K_{-1}(F)$ and the ring of $SL_2\mathbb{C}$ characters of F. Rotating the skein relation $\pi/4$ radians yields,

(20) 

Taking the difference of the two versions yields,

(21) 

Therefore crossings don't count, and $K_{-1}(F)$ is a commutative algebra. The first fundamental theorem of classical invariant theory implies that the $SL_2\mathbb{C}$-character ring of $\pi_1(F)$ is spanned by functions that are the trace of conjugacy classes in $\pi_1(F)$. The second fundamental theorem says that all relations between those functions come from the relations above plus relations from the fundamental group of F. The most important is functional evaluation of the fully polarized Cayley-Hamilton identity that we show in Figure 4

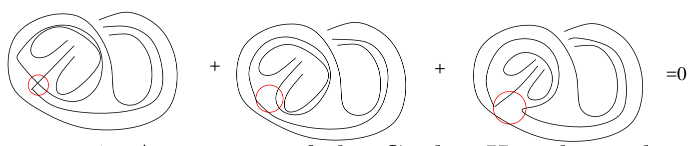


FIGURE 4. A portrait of the Cayley Hamilton identity as a skein relation, $\eta(A)\eta(B) + \eta(AB) + \eta(A^{-1}B) = 0$.

Theorem 1. *The map $\Theta : K_{-1}(F) \to X(F)$ is an isomorphism.*

The **radical**, $\sqrt{0}$ of a commutative ring is the ideal made up of all nilpotent elements. Bullock proved that for any three-manifold $\Theta : K_{-1}(M)/\sqrt{0} \to \mathcal{X}(M)$ is an isomorphism. A year later Przytycki and Sikora gave a different proof. Charles and Marché proved that the radical of $K_{-1}(F)$, where F is a closed surface is trivial and hence $\Theta : K_{-1}(F) \to \mathcal{X}(F)$ is an isomorphism for any closed surface. Recently, Przytycki and Sikora proved that $K_A(F)$ never has zero divisors.

2.3. **The threading map.** The Chebyshev polynomials of the first kind $T_k(x)$ are defined recursively by $T_0(x) = 2$, $T_1(x) = x$ and for $k > 1$,

$$T_k(x) = xT_{k-1}(x) - T_{k-2}(x). \tag{22}$$

They are derived by requiring

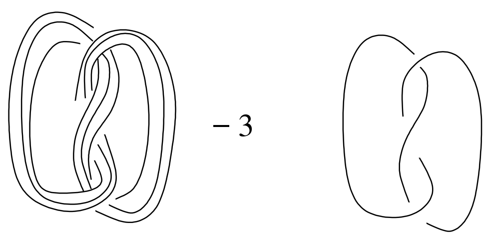
$$T_k(2\cos\theta) = 2\cos k\theta. \tag{23}$$

This means they satisfy the product to sum formula

$$T_m(x)T_n(x) = T_{m+n}(x) + T_{|m-n|}(x), \tag{24}$$

and $T_m(T_n(x)) = T_{mn}(x)$.

Given a framed link, you can thread it by $T_n(x)$ by using the annulus as a guide, and treating the operation of coloring as multlinear in the components of the link. For instance $T_3(x) = x^3 - 3x$. Threading the Trefoil with $T_3(x)$ yields:



If the link has multiple components you thread it multilinearly. If there were two components of the framed link and you were threading them both with T_3, then there would be four terms to the threaded framed link. One term where both components are cabled by three copies of themselves, minus three times two terms where one component is cabled by three copies of itself and the other component is unchanged, plus 9 times a copy of the original framed link.

Suppose that M is an oriented three-manifold and $\mathcal{L}$ is the set of framed links in M up to isotopy. For each odd n there is a map

$$\tau_n : \mathbb{C}\mathcal{L} \to \mathbb{C}\mathcal{L} \tag{25}$$

given by threading every framed link with the Nth Chebyshev polynomial of the first kind.

Theorem 2 (Bonahon, Wong)**.** *Let n be an odd counting number, and ζ a primitive $2n$th root of unity. The threading map descends,*

$$\tau_n : K_{-1}(M) \to K_\zeta(M). \tag{26}$$

Furthermore if a component of a link has been threaded by T_n then you can arbitrarily change crossings involving that component and not change the the skein it represents in $K_\zeta(M)$.

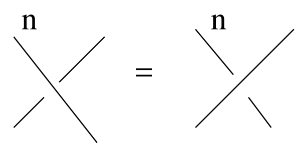


FIGURE 5. A crossing involving component threaded with T_n can be changed without changing the skein

Recall that if A is an algebra, the center of A, denoted $Z(A)$ is the the set of all elements that commute with everything.

$$Z(A) = \{z | \forall a \in A \ za = az\}. \tag{27}$$

Suppose that F is an oriented finite type surface. That means there is a closed oriented surface $\hat{F}$ and finitely many points p_i so that $\hat{F} - \{p_1, \ldots, p_n\} = F$. If F is closed, $\hat{F} = F$. Let ∂_i be the skein induced by the simple diagram that bounds a punctured disk about p_i.

Theorem 3 (Frohman-Kania-Bartoszynska-Lê). *If ζ is a primitive $2n$th root of unity, and F is a finite type surface then*

$$Z(K_\zeta(F)) = \tau_n(K_{-1}(F))[\partial_1, \ldots, \partial_n]. \tag{28}$$

If F is closed then the center is exactly the image of the threading map.

2.4. **Parametrizing simple diagrams and the trace.** An ideal triangle is a triangle with its vertices removed. An ideal triangulation of a finite type surface F is a collection of ideal triangles $\{\Delta_i\}_{i \in I}$ with an identification of their sides in pairs to get a topological space X, along with a homeomorphism $h : X \to F$. Alternatively you can think of an ideal triangulation as a collection of lines E properly embedded in F that cut F into a collection of ideal triangles.

Suppose that the finite type surface admits an ideal triangulation with T triangles and E edges.If you think about Euler characteristic, $vertices - edges + faces$, since there are no vertices,

$$\chi(F) = T - E. \tag{29}$$

Since each triangle has three edges, but each edge is shared by two triangles,

$$T = \frac{2}{3}E. \tag{30}$$

Thus $\chi(F) = -\frac{1}{3}E$, or the number of edges is $-3\chi(F)$. Among other things F can have an ideal triangulation if and only if it has negative Euler characteristic.

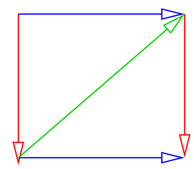


FIGURE 6. An ideal triangulation of the once punctured torus. Identify edges of the same color according to the arrows.

A **folded triangle** is a triangle that has had two of its edges identified in an ideal triangulation. It is always possible to avoid fold triangles, so we always assume the the triangles in our ideal triangulations are embedded.

If α and β are two properly embedded one manifolds in a surface and at least one is compact, then the **geometric intersection number** of α and β, denoted $i(\alpha,\beta)$ is the minimimum cardinality of $\alpha' \cap \beta'$ where α' and β' are isotopic to α and β by a compactly supported isotopy and α' and β' are transvserse.

We say that α and β realize their geometric intersection number and the cardinality of $\alpha \cap \beta$ is $i(\alpha,\beta)$. We say α and β **form a bigon** if there is a disk D embedded in F so that $\partial D = a \cup b$ where $a \subset \alpha$ and $b \subset \beta$ and $D \cap \alpha = a$, and $D \cap \beta = b$.

Suppose that F has an ideal triangulation with edges E. A simple diagram S is said to be in **normal position** with respect to the triangulation if it forms no bigons with any of the edges. Let $f_S : E \to \mathbb{N}$ be defined by

$$f_S(e) = i(S,e). \tag{31}$$

If S is in normal position then the cardinality of $S \cap e$ is equal to $f_S(e)$. An analysis of isotopy classes of proper system of arcs in an ideal triangle shows that two diagrams S and S' are isotopic if and only if $f_S = f_{S'}$.

Not every function comes from a simple diagram. If a,b,c are the sides of a triangle, and S is a simple diagram then $f_S(a)+f_S(b)+f_S(c)$ is even as a compact one manifold has an even number of endpoints. A corner of a triangle is determined by the choice of two sides. Diagram in normal position intersects an ideal triangle in arcs that have their endpoints in two sides. The number of arcs having their endpoints in a pair of sides is called a **corner number**. You can compute the corner numbers as,

$$c(\{a,b\}) = \frac{f(a)+f(b)-f(c)}{2}, c(\{a,c\}) = \frac{f(a)+f(c)-f(b)}{2}, \tag{32}$$

and

$$c(\{b,c\}) = \frac{f(b)+f(c)-f(a)}{2}. \tag{33}$$

In order to correspond to a diagram these numbers should all be greater or equal to zero. A function $f : E \to \mathbb{N}$ is said to be an **admissible coloring** if whenever a, b, c are the sides of an ideal triangle then $f(a)+f(b)+f(c)$ is even and $f(a), f(b), f(c)$ satisfy all triangle inequalities.

Theorem 4. *There is a one to one correspondence between isotopy classes of simple diagrams on the surface F and admissible colorings of an ideal triangulation of F.*

The admissible colorings of an ideal triangulation form a pointed integral cone under addition. An admissible coloring f is said to be **indivisible** if whenever $f = f_1 + f_2$ where f_1 and f_2 are admissible colorings then $f_1 = 0$ or $f_2 = 0$. It is classical theorem that every pointed integral cone has finitely many indivisible elements, and they are the unique additive generating set of minimal cardinality.

In the case of the once punctured torus, representing the admissible colorings as three-tuples of nonnegative integers, the indivisible colorings are $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$. These correspond to the longitude, meridian, and a $(1,1)$-curve on the punctured torus.

Choose an ordering of E. Use this to order $\mathbb{N}^E$ lexicographically. Notice that $\mathbb{N}^E$ in the lexicographic ordering is a well ordered monoid. By that we mean $\mathbb{N}^E$ is well ordered and if $a, b \in \mathbb{N}$ have $a < b$ then for any $c \in \mathbb{N}$, $a + c < b + c$. Since $\mathcal{A}$ is a submonoid of $\mathbb{N}^E$ we have that $\mathcal{A}$ is a well ordered monoid.

If $\alpha \in K_\zeta(F)$ then we can write α as a finite linear combination of simple diagrams with complex coefficients,

$$\alpha = \sum_S z_S S \tag{34}$$

where the S are simple diagrams, and the z_S are nonzero elements of $\mathfrak{D}$. The **lead term** of α is $z_S S$ where S is the largest diagram appearing in the sum. We denote the lead term of the skein α as $ld(\alpha)$.

Theorem 5 (Abdiel-Frohman)**.** *Let F be a finite type surface with negative Euler characteristic and at least one puncture. Let E be the* *edges of an ideal triangulation and assume an ordering on E. Let $\alpha, \beta \in K_\zeta(F)$ be nonzero. Suppose $ld(\alpha) = zS$, and $ld(\beta) = z'S'$, and $f_S, f_{S'} : E \to \mathbb{N}$ are the colorings corresponding to S and S'. Let S'' be the simple diagram with coloring $f_S + f_{S'}$. There exists $k \in \mathbb{Z}$ so that The lead term of $\alpha * \beta$ is $\zeta^k zz'S''$.*

Which quickly leads to:

Theorem 6 (Abdiel-Frohman)**.** *Let F be a finite type surface with ideal triangulation having edge set E. Suppose that $S_1, \ldots, S_k$ are the simple diagrams corresponding to the indivisible admissible colorings of E. The skeins*

$$S_1^{j_1} * \ldots * S_k^{j_k} \tag{35}$$

where the j_i range over all counting numbers spans $K_\zeta(F)$.

An algebra is **affine** if it finitely generated. This proves that $K_\zeta(F)$ is affine.

Theorem 7 (Abdiel-Frohman)**.** *Let F be a finite type surface with ideal triangulation having edge set E. Suppose that $S_1, \ldots, S_k$ are the simple diagrams corresponding to the indivisible admissible colorings of E. The skeins*

$$S_1^{j_1} * \ldots * \ldots S_k^{j_k} \tag{36}$$

where the j_i range over $\{0, \ldots, n-1\}$ spans $K_\zeta(F)$ as a module over $Z(K_\zeta(F)$.

This means that $K_\zeta(F)$ has finite rank as a module over $Z(K_\zeta(F))$.

Theorem 8 (Muller, Charles-Marcheé, Przytycki-Sikora, Frohman-Kania-Bartoszynska)**.** *The algebra $K_\zeta(F)$ has no zero divisors.*

Since $K_\zeta(F)$ has no zero divisors $S = Z(K_\zeta(F)) - \{0\}$ is a multiplicatively closed subset of the center that does not contain 0. We can localize to make every element of S invertible. This means that $S^{-1}K_\zeta(F)$ is an finite dimensional algebra over the field $S^{-1}Z(K_\zeta(F)$.

Suppose that F is a finite type surface with ideal triangulation having edges E. Given a simple diagram S we can reduce the admissible coloring $f_S : E \to \mathbb{N}$ modulo n to get $(f(e_1), \ldots, f(e_k)) \in \mathbb{Z}_n^E$. Given a skein α, it has lead term $ld(\alpha) = zS$, the reduction of the admissible coloring of S modulo n is the **residue** of the skein, denoted $res(\alpha)$.

Theorem 9 (Frohman-Kania-Bartoszynska)**.** *A set of skeins $\mathcal{B}$ in $K_\zeta(F)$ forms a basis for $S^{-1}K_\zeta(F)$ over $S^{-1}\tau_n(K_{-1}(F))$ if and only the set of residues of the skeins in $\mathcal{B}$ consists of exactly all the elements of $\mathbb{Z}_n^E$ without repitition.*

Theorem 10 (Frohman-Kania-Bartoszynska)**.** *Suppose that F is a finite type surface with p punctures, and Euler characteristic $\chi(F)$. The dimension of $S^{-1}K_\zeta(F)$ over $S^{-1}Z(K_\zeta(F))$ is $n^{-3\chi(F)-p}$.*

There is a trace, $tr : K_\zeta(F) \to \tau_n(K_{-1}(F))$ that is $Z(K_\zeta(F)$-linear. Since $S^{-1}K_\zeta(F)$ is a finite dimensional vector space over $S^{-1}\tau_n(K_{-1}(F))$ if $\alpha \in K_\zeta(F)$ then there is a $S^{-1}\tau_n(K_{-1}(F))$-linear map

$$L_\alpha : S^{-1}K_\zeta(F) \to S^{-1}K_\zeta(F) \tag{37}$$

given by $L_\alpha(\beta) = \alpha * \beta$. The dimension of $S^{-1}K_\zeta(F)$ as a vector space over $S^{-1}\tau_n(K_{-1}(F))$ is $n^{-3\chi(F)}$. Let

$$tr(\alpha) = \frac{1}{n^{-3\chi(F)}} Tr(L_\alpha). \tag{38}$$

The amazing thing is that to define the trace we needed to localize, and yet the trace is well defined as a map on the unlocalized algebras.

Given a special basis of $K_\zeta(F)$ there is an easy computation of the trace. Recall, a simple diagram is primitive is no two curves in the diagram are parallel. Suppose that P is a primitive diagram with components J_i. Choose positive integers k_i for each i. The skein

$$\prod_i T_{k_i}(J_i) \tag{39}$$

is a threaded primitive diagram. Since the lead terms of threaded primitive diagrams can be place in one to one correspondence with simple diagrams, they form a basis for $K_\zeta(F)$ over the complex numbers. To compute $tr(\alpha)$, write α as a linear combination of threaded primitive diagrams, and then strike out any term, where any of the threading indices k_i of the diagram is not divisible by n.

The trace tr is nondegenerate in the sense that if $\alpha \in K_\zeta(F)$ is not zero there exists $\beta \in K_\zeta(F)$ so that $tr(\alpha * \beta) \neq 0$. It is cyclic in the sense that $tr(\alpha * \beta) = tr(\beta * \alpha)$, and $tr(tr(\alpha)) = tr(\alpha)$ so it is a projection. This means $K_\zeta(F)$ is a **Cayley-Hamilton algebra**. A consequence is that the equivalence classes of representations of $K_\zeta(F)$ is naturally an algebraic set that can be described more or less formally [16] .

Theorem 11 (Frohman-Kania-Bartoszynska)**.** *$S^{-1}K_\zeta(F)$ is a division algebra.*

That means every nonzero element has a multiplicative inverse. Schur's lemma says that the commutant of an irreducible representation that takes on values in $M_n(k)$ where k is a field, is a division algebra over k.

Conjecture 1. *There is an irreducible, projective representation of the mapping class group defined over the function field of the character variety of a finite type surface F, so that the commutant of the representation is $S^{-1}K_\zeta(F)$.*

3. Lecture Ib: Representation Theory of Algebras

3.0.1. *Algebras.* An **algebra** A over the field $\mathbb{C}$ is a vector space A over $\mathbb{C}$ along with a $\mathbb{C}$-bilinear associative multiplication, that has a unit element 1. We denote multiplication by juxtaposition. The unit element 1 is characterized by the property that for all $a \in A$, $1a = a1 = a$.

For example $M_n(\mathbb{C})$ the $n \times n$ matrices with complex entries are an algebra over the complex numbers, where the product comes from matrix multiplication. The identity element is the $n \times n$ identity matrix Id_n. Let $E_{i,j}$ denote the $n \times n$ matrix all of whose entries are zero except for the entry in the ith row and jth column which is 1. These form a basis for $M_n(\mathbb{C})$ over the complex numbers, and

$$E_{i,j}E_{k,l} = \delta_j^k E_{i,l} \tag{40}$$

where δ_j^k is the Kronecker delta.

We say that $u \in A$ is a **unit** if there exists $v \in A$ with $uv = vu = 1$. A matrix A is a unit in $M_n(\mathbb{C})$ if and only if its determinant is nonzero.

The **center** of an algebra A, denoted $Z(A)$ is the subalgebra of all $z \in A$ so that for all $a \in A$, $za = az$. If A is commutative then $Z(A) = A$. If $A = M_n(\mathbb{C})$ then $Z(A)$ is all scalar multiples of the identity matrix .

A **homomorphism** $\phi : A \to B$ of algebras has the properties that $\phi(1) = 1$ and for all $a_1, a_2 \in A$, $\phi((a_1a_2) = \phi(a_1)\phi(a_2)$.

An algebra is **affine** if there are elements $x_1, \ldots, x_n$ in A so that every element of A can be written as a linear combination of monomials in the x_i This is equivalent to saying the algebra is a quotient of the free algebra $\mathbb{C} < x_1, \ldots, x_n >$ of noncommutative polynomials in the variables x_i.

The **Artin-Tate Lemma** says that if A is an affine algebra , and has finite rank as a module over its center $Z(A)$ then $Z(A)$ is an affine algebra.

3.0.2. *Ideals.* A **two-sided ideal** $I \leq A$ is a vector subspace of A so that if $a \in A$ and $h \in I$, then $ah \in I$ and $ha \in I$. If $\rho : A \to B$ is a homomorphism of algebras then $ker(\rho)$ is a two sided ideal.

3.1. **Central Simple Algebras.** The only two sided ideals of $M_n(\mathbb{C})$ are the trivial ideal $\{0\}$ and $M_n(\mathbb{C})$. We say that A is **central simple** if it has no nontrivial two sided ideals and its center is exactly complex multiples of the identity. Hence $M_n(\mathbb{C})$ is central simple. In fact it is a consequence of the Artin-Wedderburn theorem that a central simple algebra over the complex numbers that has finite dimension is a matrix algebra.

In a more mature exposition, algebras can be defined over any field, not just the complex numbers. A division algebra D is an algebra so that every nonzero element has a multiplicative inverse. The **Artin-Wedderburn theorem** *says that if A is a finite dimensional central simple algebra over the field k, then there is a division algebra D over k and an integer n so that A is isomorphic to $M_n(D)$ the algebra of $n \times n$ matrices with coefficients in D. The complex numbers are the only division algebra over the complex numbers. Hence matrix algebras are the only central simple algebras over $\mathbb{C}$.*

Suppose $k \leq E$ is a finite field extension, that is E is a field, and E is a finite dimensional vector space over k. Suppose further that A is a finite dimensional algebra over k. We can form,

$$A \otimes_k E \tag{41}$$

which is now a finite dimensional algebra over E. We say $A \otimes_k E$ is the result of extending the coefficients of A. The reason is that if $\{v_i\}$ is a basis of A over k then $\{v_i \otimes 1\}$ is a basis for $A \otimes_k E$ over E. You can just treat $A \otimes_k E$ as having the same basis as A, but with the coefficients of that basis coming from E. The center of $A \otimes_k E$ is $Z(A) \otimes_k E$.

If D is a finite dimensional division algebra over a field k then the dimension of D as a vector space is n^2 for some n. It is possible extend the coefficients of D to some finite extension E of k so that the extended algebra is isomorphic to $M_n(E)$. Obviously, $D \otimes_k E = M_n(E)$ is not a division algebra.

For instance the quaternions, $\mathbb{H}$, are a 4 dimensional vector space over $\mathbb{R}$. Recall that $\mathbb{H}$ is a four dimensional vector space over $\mathbb{R}$ with basis $\{1, i, j, k\}$ where 1 is the identity, and relations

$$ij = k, jk = i, ki = j, i^2 = j^2 = k^2 = -1. \tag{42}$$

From the relations it is easy to see that the center of $\mathbb{H}$ is exactly real scalar multiples of the identity.

The complex numbers are a degree two extension of the reals. Extending coefficients to $\mathbb{C}$, every element of

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \tag{43}$$

can be written as a complex linear combination

$$\alpha 1 + \beta i + \gamma j + \delta k. \tag{44}$$

Define a homomorphism

$$\theta : \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \to M_2(\mathbb{C}) \tag{45}$$

by sending 1 to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, i to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, j to $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and k to $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. It is easy to see that the four matrices are linear independent over $\mathbb{C}$ and satisfy the defining equations of the quaternions. Hence $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $M_2(\mathbb{C})$.

3.2. **Prime algebras.** An algebra A is prime if for any $a, b \in A$ if it is the case that for all $r \in A$, $arb = 0$ then either $a = 0$ or $b = 0$.

If A is commutative this is equivalent to saying that if $ab = 0$ then $a = 0$ or $b = 0$. A prime commutative algebra is an **integral domain**.

The algebra $M_n(\mathbb{C})$ is prime. If $a, b \in M_N(\mathbb{C})$ we can write them uniquely as complex linear combinations of the $E_{i,j}$. That is,

$$a = \sum_{i,j} a^{i,j} E_{i,j} \text{ and } b = \sum_{k,l} b^{k,l} E_{k,l}. \tag{46}$$

The assumption that $a, b \neq 0$ means that there is $a^{m,n} \neq 0$ and $b^{r,s} \neq 0$. Notice that

$$aE_{n,r}b = \sum_{i,l} a^{i,n} b^{r,l} E_{i,l}. \tag{47}$$

we know for a fact that $a^{m,n}b^{r,s} \neq 0$, and the $E_{i,l}$ are linearly independent. Hence, the product is nonzero. Therefore $M_n(\mathbb{C})$ is a prime algebra.

If A is an algebra and $I \leq A$ is a two sided ideal we say that I is **prime** if whenever $arb \in I$ for all $r \in A$ then either $a \in I$ or $b \in I$. This is equivalent to A/I being a prime algebra.

3.2.1. *Localization.* Suppose that A is an algebra and $S \leq Z(A)$ is **multiplicatively closed**. That is if $s_1, s_2 \in S$ then $s_1 s_2 \in S$. If $0 \notin S$ then we can form the **localization** of A at S. Start with ordered pairs $(a, s) \in A \times S$. We say $(a, s) \sim (b, t)$ if there is a unit $u \in S$ so that $uta = ubs$. In the case that the center of A is an integral domain, this can be simplified to $(a, s) \sim (b, t)$ if $ta = bs$. In the cases we work with the center is always an integral domain, so we don't need u. Let $[a, s]$ denote the equivalence class of (a, s) under this relation. Define addition and multiplication by

$$[a, s] + [b, t] = [at + bs, st] \; [a, s][b, t] = [ab, st]. \tag{48}$$

Denote the quotient space by $S^{-1}A$ it is an algebra over $\mathbb{C}$. There is a homomorphism $\iota : A \to S^{-1}A$ given by $\iota(a) = [a, 1]$. Notice that the image of every element of S is a unit in $S^{-1}A$ because $[s, 1][1, s] = [1, 0]$. The map ι is not necessarily injective, but if A is prime then it is.

Proposition 1. *If A is a prime algebra, $S \subset Z(A)$ a multiplicatively closed subset that does not contain 0, then the map $\iota : A \to S^{-1}A$ given by $\iota(a) = [a, 1]$ is injective.*

Proof. Suppose that $\iota(a) = [a, 1] = [0, s]$. That means $as = 0$. Since s is central, for any $r \in A$, $asr = ars$ is zero. Since A is prime that implies that $a = 0$ or $s = 0$. Since $0 \notin S$, it must be that $a = 0$. □

3.3. **Representations.** If A is an algebra, then a **left A-module**, or a **representation of A**, is a vector space V along with homomorphism $\rho : A \to Lin(V)$ into the $\mathbb{C}$-linear maps from V to itself.

We restrict our attention to modules that are finite dimensional vector spaces. It is traditional to suppress ρ so that if $a \in A$ the result of applying $\rho(a)$ to the vector v is denoted

$$a.v. \tag{49}$$

The statement that ρ is a homomorphism in this notation is equivalent to the following two statements;

- $1.v = v$, and,

- $a.(b.v) = (ab).v$.

If V is finite dimensional we can choose a basis $v_1, \ldots, v_n$. If $L \in Lin(V)$ then for all j,

$$L(v_j) = \sum_i a_{i,j} v_i. \tag{50}$$

This defines a isomorphism $B : Lin(V) \to M_n(\mathbb{C})$ given by $L \to (a_{ij})$. This allows us to think of finite dimensional left modules as linear actions of A column vectors $\mathbb{C}^n$, and the associated homomorphism to have range contained in $n \times n$ matrices. That is $\rho : A \to M_n(\mathbb{C})$.

The representation $\rho : A \to Lin(V)$ is **irreducible** if one of the following equivalent properties holds;

- If $W \leq V$ is a vector subspace of V and $A.W \leq W$ then $W = \{0\}$ or $W = V$.
- If $v \neq 0 \in V$ then $A.v = V$. (This is often described by saying that V is **strongly cyclic**, in the sense that it is the cyclic module on any nonzero element.)
- The associated homomorphism $\rho : A \to Lin(V)$ is onto. This is a consequence of the more general theorem called the **Jacobson Density Theorem**.

Two left modules V and W are **equivalent** if there is a linear isomorphism $L : V \to W$ so that for any $a \in A$, $a.L(v) = L(a.v)$. In terms of homomorphisms, if $\rho_1 : A \to Lin(V)$ and $\rho_2 : A \to Lin(W)$ are the homomorphisms corresponding to the two modules, then for all $a \in A$, $L^{-1}\rho_1(a)L = \rho_2(a)$.

In the algebraic view of geometry, equivalence classes of irreducible representations of an algebra are the points of the geometric object associated to that algebra.

If C is a commutative algebra, then the only irreducible representations of C are one dimensional. Two one dimensional representations are equivalent if and only if they are equal. Hence an irreducible representation of C is a homomorphism $\phi : C \to \mathbb{C}$. The kernel of ϕ is a maximal ideal, and by the **weak nullstellensatz** that maximal ideal determines ϕ. Let $Max\ Spec(C)$ denote the set of maximal ideals of C. Define a topology on $Max\ Spec(C)$ using the subbasis of all

$$S_c = \{\mathfrak{m} \in Max\ Spec(C) | c \notin \mathfrak{m}\}. \tag{51}$$

This is called the Zariski topology.

3.3.1. *Skolem-Noether Theorem.* An **automorphism** $\theta : A \to A$. of the algebra A is a one to one and onto homomorphism of A to itself. One way to construct automorphisms of an algebra is to conjugate by a unit. Let $C \in A$ be a unit. We define

$$\Theta_C : A \to A \tag{52}$$

by $\Theta_C(a) = C^{-1}aC$. The map Θ is one to one and onto as its inverse is $\Theta_{C^{-1}}$. It is a homomorphism because

$$\Theta_C(a_1a_2) = C^{-1}a_1a_2 = C^{-1}a_1CC^{-1}a_2C = \Theta_C(a_1)\Theta_C(a_2), \tag{53}$$

and $\Theta_C(1) = 1$. We call such automorphisms **inner automorphisms**.

The **Skolem-Noether theorem** says that every automorphism of $M_n(\mathbb{C})$ is inner.

Theorem 12. *If ρ_1, ρ_2 are irreducible representations of A having kernels I_1 and I_2, then ρ_1 is equivalent to ρ_2 if and only if $I_1 = I_2$.*

Proof. First suppose that ρ_1 and ρ_2 are equivalent. This means there is n so that

$$\rho_1, \rho_2 : A \to M_n(\mathbb{C}) \tag{54}$$

and there is an invertible $L \in M_n(\mathbb{C})$ so that for all $a \in A$,

$$L^{-1}\rho_1(a)L = \rho_2(a). \tag{55}$$

This means that $\rho_1(a) = 0$ if and only if $\rho_2(a) = 0$, so $I_1 = I_2$.

Now assume that $I_1 = I_2$. By the first isomorphism theorem the ρ_i induces isomorphisms

$$\overline{\rho}_1 : A/I_1 \to M_n(C), \text{ and } \overline{\rho}_2 : A/I_2 \to M_n(\mathbb{C}). \tag{56}$$

Since $I_1 = I_2$, $\overline{\rho}_2 \circ \overline{\rho}_1^{-1} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is an automorphism of $M_n(\mathbb{C})$. By the Skolem-Noether theorem there exists $L \in M_n(\mathbb{C})$ so that for all matrices M, $L^{-1}ML = \overline{\rho}_2 \circ \overline{\rho}_1^{-1}(M)$. If $M = \rho_1(a)$ for $a \in A$ then

$$L^{-1}\rho_1(a)L = \rho_2(a). \tag{57}$$

Therefore ρ_1 and ρ_2 are equivalent. $\square$

3.4. **The Skolem-Noether Theorem for mortals.** Given $\theta : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ how do you find the matrix L so that for all matrices A,

$$\theta(A) = L^{-1}AL? \tag{58}$$

To do this you just need to understand what conjugation looks like. We will keep things small. Suppose that L is a 2×2 matrix with determinant 1, say

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{59}$$

From the cofactor formula for the inverse we know,

$$L^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{60}$$

Notice that

$$L^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}L = \begin{pmatrix} ad & bd \\ -ac & -bc \end{pmatrix}. \tag{61}$$

Also,

$$L^{-1}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}L = \begin{pmatrix} -ab & -b^2 \\ a^2 & ab \end{pmatrix}. \tag{62}$$

Notice that if $a \neq 0$ then the first columns of these matrices are a times the columns of L^{-1}. Since L is invertible some entry in its first row is nonzero, say the jth. The matrix whose ith column is the jth column of $L^{-1}E_{i1}L$ is a nonzero scalar multiple of L^{-1}.

Proposition 2. *If $\theta : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is an automorphism. Choose j so that the jth column of $\theta(E_{11})$ is not the zero vector. The matrix L^{-1} that has the jth column of $\theta(E_{i1})$ as its ith column has the property that for all $A \in M_n(\mathbb{C})$,*

$$\theta(A) = L^{-1}AL. \tag{63}$$

□

3.4.1. *The central character of a representation.* If $\rho : A \to B$ is an onto algebra homomorphism then the image of the center of $Z(A)$ under ρ is contained in $Z(B)$. Therefore if $\rho : A \to M_n(\mathbb{C})$ is an irreducible representation and $z \in Z(A)$ then

$$\phi(z) = \chi_\rho(z)Id_n, \tag{64}$$

where $\chi_\rho(z) \in \mathbb{C}$ and Id_n is the $n \times n$ identity matrix. The map

$$\chi_\rho : Z(A) \to \mathbb{C} \tag{65}$$

is called the **central character** of the representation ρ.

This is a good time to reflect on equivalence of representations. We have seen that two irreducible representations are equivalent if they have the same kernels. If I is the kernel of the representation ρ, then the kernel of χ_ρ is $I \cap Z(A)$. On the other hand, the weak nullstellensatz tells us that if $Z(A)$ is affine, then central characters are classified by their kernels. Hence representations are classified by their central character if and only if the kernels of the representations are determined by their intersection with the center. There is a class of algebras for which this is true.

3.5. **Azumaya Algebras.** If A is an algebra, then you can view A as a module over $Z(A)$. Let $End_{Z(A)}(A)$ be the algebra of all maps from A to A that are $Z(A)$ linear. Let A^{op} denote A with the opposite multiplication. That is when we write ab we mean ba. There is a map from $\Psi : A \otimes_{Z(A)} A^{op} \to End_{Z(A)}(A)$ given by

$$\Psi(a \otimes b)(c) = acb. \tag{66}$$

An algebra A is **Azumaya** if it is a finite rank projective module over it's center, and the map Ψ is an isomorphism of algebras.

If A is Azumaya and I is any two sided ideal then

$$I = (I \cap Z)A, \tag{67}$$

that is if two two sided ideals I_1 and I_2 have the same intersection with the center then they are the same ideal.

Theorem 13. *Irreducible representations of Azumaya algebras are classified by their central characters.*

□

The Azumaya condition is so strong that you cannot reasonably expect a naturally defined algebra to have it. However, there are very general situations where you can localize an algebra so that it becomes Azumaya.

Theorem 14 (Posner)**.** *Let A be a prime affine k-algebra that has finite rank over its center $Z(A)$. Let $S = Z(A) - \{0\}$. The algebra $S^{-1}A$ is central simple over $S^{-1}Z(A)$.*

□

By extending coefficients to a finite extension E of the center of $S^{-1}A$,

$$S^{-1}A \otimes_{S^{-1}Z(A)} E = M_n(E) \tag{68}$$

We call n the dimension of A. Posner's theorem says that there is an embedding $\eta : A \to M_n(E)$ so that $AZ(M_n(E)) = M_n(E)$. That means that every element of A can be written as a matrix with coefficients from E. It also means that as a vector space over $S^{-1}Z(A)$, the dimension of $S^{-1}A$ is n^2. Using deep theorems about matrix algebras, both Artin, and Procesi proved:

Theorem 15. *If A is a prime affine k-algebra, that has finite rank over its center, then there exists $c \in Z(A)$ so that if $S = \{c^k | k \in \mathbb{N}\}$ then $S^{-1}A$ is Azumaya. Furthermore, all irreducible representations of $S^{-1}A$ have dimension n.*

The slogan is that you invert a nonzero element of the Formanek center of A.

Finally,

Theorem 16. *Suppose that A is a prime algebra, and let $\mathfrak{m} \in Z(A)$ be a maximal ideal. Let $S \subset Z(A)$ be a multiplicatively closed subset so that $S \cap \mathfrak{m} = \emptyset$. Finally suppose that there is a unique two sided ideal $I \leq S^{-1}A$ so that $I \cap Z(S^{-1}(A)) = S^{-1}\mathfrak{m}$. If $I_1, I_2 \leq A$ are prime two sided ideals with $I_j \cap S = \emptyset$, and $I_1 \cap Z(A) = I_2 \cap Z(A) = \mathfrak{m}$ then $I_1 = I_2$.*

Proof. Recall the injective homomorphism $\iota : A \to S^{-1}A$ given by $\iota(a) = [a, 1]$. If

$$\iota(I_j) = \iota(A) \cap S^{-1}I_j \tag{69}$$

then the theorem follows as by hypothesis $S^{-1}I_1 = S^{-1}I_2$.

Clearly $\iota(I_j) \leq \iota(A) \cap S^{-}I_j$. To finish we need to prove $\iota(A) \cap S^{-1}I_j \leq \iota(I_j)$. Suppose that $[a, s] \in \iota(A) \cap S^{-1}I_1$. This means that $a \in I_1$ and there exists $b \in A$ with $[a, s] = [b, 1]$. By the definition of equivalence, $bs = a$. However, I_1 is prime. Since $bs \in I_1$, and s is central, for every $r \in A$, $brs \in A$. This implies that $b \in I_1$ or $s \in I_1$. Since $S \cap I_1 = \emptyset$ this means $b \in I_1$, implies $S^{-1}I_1 \cap \iota(A) \subset \iota(I_1)$. □

If V is the maximal spectrum of the algebra A, and S is the powers of $c \in Z(A)$, so that c is not nilpotent, then $S^{-1}A$ exists and its maximal spectrum is the Zariski open subset of V,

$$V_c = \{\mathfrak{m} \in V | c \notin \mathfrak{m}\}. \tag{70}$$

Putting it all together, we get the following theorem of Frohman,Kania-Bartoszynska and Lê:

Theorem 17 (Unicity Theorem)**.** *Suppose that A is a prime affine algebra that has finite rank as a module over its center. There is a Zariski open subset V_c so that there is a unique equivalence class of irreducible representations of A for each $\mathfrak{m} \in V_c$, so that $\mathfrak{m}$ is the kernel of the central character of the representations.*

Theorem 18 (Frohman-Kania-Bartoszynska-Lê)**.** *The skein algebras $K_\zeta(F)$ where F is an oriented finite type surface having Euler characteristic $\chi(F)$ and p punctures, and ζ is a primitive nth root of unity, satisfy the hypotheses of the unicity theorem. Therefore, there is a one to one correspondence between a dense open subset of the $SL_2\mathbb{C}$-character variety of F and irreducible representations of $K_\zeta(F)$ that take on values in $M_N(\mathbb{C})$ where N is the square root of dimension of $S^{-1}K_\zeta(F)$ as a vector space over $S^{-1}Z(K_\zeta(F))$.*

4. Lecture IIa: Noncommutative Tori and Skein Algebras

4.1. **The $SL_2\mathbb{C}$-character variety of $\mathbb{Z}\times\mathbb{Z}$.** Representations of $\mathbb{Z}\times\mathbb{Z}$ into $SL_2\mathbb{C}$ are in one to one correspondence with choices of matrices $(L, M) \in SL_2\mathbb{C}^2$ that commute. If two matrices commute and are diagonable, they are simultaneously diagonal. There are extactly two conjugacy classes of nondiagonable matrices in $SL_2\mathbb{C}$,

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & \lambda \\ 0 & -1 \end{pmatrix}, \tag{71}$$

where λ is nonzero. The traces of these matrices are indistinguishable from the trace of $\pm Id_2$. In fact the closure of these two conjugacy classes includes $\pm Id_2$. Hence every representation of $\mathbb{Z}\times\mathbb{Z}$ into $SL_2\mathbb{C}$ is trace equivalent to a representation of the form,

$$\left(\begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix}, \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}\right), \tag{72}$$

where $l, m \in \mathbb{C}-\{0\}$. Let $\mathbb{C}^*$ denote $\mathbb{C}-\{0\}$. We identify $\mathbb{C}^*\times\mathbb{C}^*$ with pairs of diagonal matrices by letting (l, m) correspond to the matrices having l and m in their upper lefthand corner. If $X(T^2)$ is the character variety of the fundamental group of the torus, there is an onto mapping

$$C : \mathbb{C}^* \times \mathbb{C}^* \to X(T^2) \tag{73}$$

that takes the representation to its trace equivalence class. This mapping is a two-fold branched cover, with deck transformation,

$$\theta : \mathbb{C}^* \times \mathbb{C}^* \to: \mathbb{C}^* \times \mathbb{C}^*, \tag{74}$$

is given by $\theta(l, m) = (l^{-1}, m^{-1})$. To see this notice that if you conjugate a diagonal matrix by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{75}$$

it has the effect of permuting the two diagonal elements. There are four branch points of C corresponding to the fixed points of θ. The projection C is an algebraic mapping and it gives rise to an embedding of coordinate rings

$$C^* : C[X(T^2)] \to C[\mathbb{C}^* \times \mathbb{C}^*] \tag{76}$$

Since the first coordinate ring is isomorphic to $K_{-1}(T^2)$ and the second is $\mathbb{C}[l^{\pm 1}, m^{\pm 1}]$ we have embedded a version of the Kauffman bracket skein algebra into Laurent polynomials in two variables. The image of the embedding is exactly those functions that are fixed by the action of θ on the coordinate ring of $\mathbb{C}^* \times \mathbb{C}^*$.

This led us to believe that we could embed the skein algebra of the torus into the *noncommutative torus* [17].

4.2. **The noncommutative torus.** Let $A \in \mathbb{C} - \{0\}$. The noncommutative torus $\mathcal{W}_A = \mathbb{C}[l, l^{-1}, m, m^{-1}]_A$ is the quotient of the ring of noncommutative Laurent polynomials in l and m by the ideal generated by $lm - A^2 ml$. It is sometimes called the *exponentiated Weyl algebra.*

There is a particularly nice basis for $\mathcal{W}_A$. Let

$$e_{p,q} = A^{-pq} l^p m^q. \tag{77}$$

With respect to this basis the product has a very tractible formula,

$$e_{p,q} * e_{r,s} = A^{\begin{vmatrix} p & q \\ r & s \end{vmatrix}} e_{p+r,q+s}. \tag{78}$$

The vertical bars indicate the determinant of $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$.

From this basis it is easy to see that there is an action of $SL_2\mathbb{Z}$ on $\mathcal{W}_A$ as automorphisms.

If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z}$, define

$$M.e_{p,q} = e_{ap+bq,cp+dq}. \tag{79}$$

The formula consists of treating the index p, q as a column vector with integer entries.

There is an automorphism $\theta : \mathcal{W}_A \to \mathcal{W}_A$ of order two given by $\theta(e_{p,q}) = e_{-p,-q}$. The **symmetric part** of $\mathcal{W}_A$ denoted $\mathcal{W}_A^\theta$ is the fixed subalgebra of F.

Next assume that the variable in the definition of the noncommutative torus is a primitive $2n$th root of unity ζ where n is odd. In this case $\mathbb{C}[l, l^{-1}, m, m^{-1}]_\zeta$ has a large center.

$$Z(\mathbb{C}[l, l^{-1}, m, m^{-1}]_\zeta) =< e_{np,nq} | (p, q) \in \mathbb{Z} \times \mathbb{Z} >, \tag{80}$$

where the $<, >$ denote the linear span.

There is a central valued, central linear trace. If $f(l, m) \in \mathbb{C}[l, l^{-1}, m, m^{-1}]_\zeta$ let

$$tr(f(l, m) = \frac{1}{n^2} \sum_{i=0,j=0}^{n-1,n-1} f((\zeta^{2i} l, \zeta^{2j} m). \tag{81}$$

As a module over $Z(\mathbb{C}[l, l^{-1}, m, m^{-1}]_\zeta)$, $\mathbb{C}[l, l^{-1}, m, m^{-1}]_\zeta$ is free with basis $e_{p,q}$ where (p, q) ranges over $\mathbb{Z}_n \times \mathbb{Z}_n$. This is enough to imply that $\mathbb{C}[l, l^{-1}, m, m^{-1}]_\zeta$ is Azumaya. Hence its equivalence classes of irreducible representations are in one to one correspondence with elements of the maximal spectrum of its center. Its center is just the ring of commutative Laurent polynomials in $l^{\pm n}$ and $m^{\pm n}$. The maximal spectrum of this ring is in one to one correspondence with

$$\mathbb{C} - \{0\} \times \mathbb{C} - \{0\}. \tag{82}$$

The irreducible representations correspond to onto homomorphisms from $\mathbb{C}[l, l^{-1}, m, m^{-1}]_\zeta$ to $M_n(\mathbb{C})$.

Given $(a, b) \in \mathbb{C} - \{0\} \times \mathbb{C} - \{0\}$ we define an action of $\mathbb{C}[l, l^{-1}, m, m^{-1}]_\zeta$ on $\mathbb{C}^n$. We index the standard basis for $\mathbb{C}^n$ by $\vec{e}_i$ where i ranges from 0 to $n - 1$. Let $x\mathbb{C}$ with $x^n = b$. Let

$$\rho(m).\vec{e}_i = \zeta^{-2i} \vec{e}_i. \tag{83}$$

Let

$$\rho(l)\vec{e}_i = \vec{e}_{i+1} \tag{84}$$

for $i < n - 1$ and $\rho(l)\vec{e}_{n-1} = a\vec{e}_0$. In the case where $n = 3$ the matrices look like

$$\rho(l) = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \ \rho(m) = \begin{pmatrix} x & 0 & 0 \\ 0 & \zeta^{-2}x & 0 \\ 0 & 0 & \zeta^{-4}x \end{pmatrix} \tag{85}$$

To see that this representation is irreducible note that

$$\frac{1}{nx}\sum_{i=0}^{n-1}\rho(m^i) \tag{86}$$

is the matrix E_{11}. You can get all the other E_{ij} by premultiplying and post multiplying by powers of $\rho(l)$ and sometimes dividing by a.

The symmetric part of $\mathbb{C}[l,l^{-1},m,m^{-1}]_\zeta$ is an order in the sense that

$$\mathbb{C}[l,l^{-1},m,m^{-1}]_\zeta^\theta Z(\mathbb{C}[l,l^{-1},m,m^{-1}]_\zeta) = \mathbb{C}[l,l^{-1},m,m^{-1}]_\zeta. \tag{87}$$

For a proof see [1]. Hence the restriction of any irreducible representation of $\mathbb{C}[l,l^{-1},m,m^{-1}]_\zeta$ to $\mathbb{C}[l,l^{-1},m,m^{-1}]_\zeta^\theta$ is still irreducible.

4.3. **The skein algebra of the torus.** A simple diagram on the torus consists of a collection of parallel curves. Oriented simple closed curves on the torus correspond to to $(p,q)\in\mathbb{Z}\times\mathbb{Z}$ so that p and q are relatively prime. If (p,q) have greatest common divisor d then you can think of (p,q) as d copies of the the oriented curve $(p/d,q/d)$. The skein algebra has as basis unoriented simple diagrams, for this reason we identify the pairs (p,q) and $(-p,-q)$. The primitive diagrams correspond to pairs that are relatively prime.

The skein algebra of the torus was presented as an algebra by Bullock and Przytycki [12]. Let x_1 and x_2 be two simple closed curves on the torus that intersect in a single point of transverse intersection. As in the example in lecture Ia, the product of the skeins corresponding to x_1 and x_2 can be resolved,

$$x_1x_2 = Ax_3 + A^{-1}z, \tag{88}$$

where x_3 and z are skeins coming from simple closed curves. $K_A(T^2)$ is generated by x_1, x_2, x_3 with relations

$$\begin{aligned} Ax_1x_2 - A^{-1}x_2x_1 &= (A^2 - A^{-2})x_3 \\ Ax_2x_3 - A^{-1}x_3x_2 &= (A^2 - A^{-2})x_1 \\ Ax_3x_1 - A^{-1}x_1x_3 &= (A^2 - A^{-2})x_2. \end{aligned} \tag{89}$$

The three curves x_1, x_2, x_3 are the vertices at infinity of an ideal triangle in Fairy diagram. The proof that these curves generate, and the relations suffice are proved by induction on complexities based on the combinatorics of the Fairy diagram.

Recall the Chebyshev polynomials of the first kind $T_0(x) = 2$, $T_1(x) = x$ and $T_k(x) = xT_{k-1}(x) - T_{k-2}(x)$. We use the basis of $K_\zeta(T^2)$ made up of threaded primitive diagrams. That means $(0,0)_c$ is 2 times the empty skein, and if $d = gcd(p,q)$ then $(p,q)_c = T_d((p/d, q/d))$. This basis has the property that

$$(p,q)_c * (r,s)_c = A^{\begin{vmatrix} p & q \\ r & s \end{vmatrix}}(p+r, q+s)_c + A^{-\begin{vmatrix} p & q \\ r & s \end{vmatrix}}(p-r, q-s)_c. \tag{90}$$

We now describe an embedding of

$$C : K_A(T^2) \to \mathbb{C}[l, l^{-1}, m, m^{-1}]_A. \tag{91}$$

Given a simple closed curve, that is (p,q) where (p,q) are relatively prime let

$$C((p,q)_c) = -e_{p,q} - e_{-p,-q}. \tag{92}$$

Thats it. The way the proof goes is, first define it for $(1,0)_c, (0,1)_c, (1,1)_c$. By the presentation of the skein algebra of the torus this defines a homomorphism. Next by induction, using the properties of the Chebyshev polynomials of the first kind, derive the formula given above [17].

The image of C is the symmetric part of the skein algebra.

The mapping class group of the torus is $SL_2\mathbb{Z}$. Its action on $K_A(T^2)$ is given by treating $(p,q)_C$ as a column vector. The map C intertwines the action of the mapping class group of the torus with the action of $SL_2\mathbb{Z}$ on $\mathcal{W}_A$.

Let ζ be a primitive $2n$th root of unity. The map, C induces

$$C^* : Rep(\mathbb{C}[l, l^{-1}, m, m^{-1}]_\zeta) \to Rep(K_\zeta(T^2)). \tag{93}$$

let $\rho : \mathbb{C}[l, l^{-1}, m, m^{-1}] \to M_n(\mathbb{C})$ be a representation, then

$$C^*(\rho) = \rho \circ C \tag{94}$$

It is easy to check that on irreducible representations, this map is $2-1$ and takes irreducible representations to irreducible representations. Hence to compute the quantum hyperbolic invariant of a mapping class of the torus with respect to a fixed representation, we can work completely in the representations of the noncommutative torus.

4.4. **The noncommutative A-polynomial.** Let $K \subset S^3$ be a knot and let M_k be the complement of a regular neighborhood of K. The manifold M_K has a torus T^2 as boundary. Placing the basepoints for the fundamental groups of M_K and ∂M_K at the same point on the peripheral torus $T^2 = \partial M_K$, if K is nontrivial we have an injective map,

$$i : \pi_1(T^2) \to \pi_1(M_k). \tag{95}$$

This in turn defines a map

$$i^* : Rep(\pi_1(M_k), SL_2\mathbb{C}) \to Rep(\pi_1(T^2), SL_2\mathbb{C}) \tag{96}$$

by restriction. If $\rho : \pi_1(M_K) \to SL_2\mathbb{C}$ then

$$i^*\rho(\alpha) = \rho(i_{\#}\alpha). \tag{97}$$

Passing to character varieties, we have a map,

$$\iota^* : X(M_K) \to X(T^2). \tag{98}$$

Considerations based on Serre duality imply that the image of ι^* is an algebraic curve. Taking the inverse image under $C : \mathbb{C}^* \times \mathbb{C}^* \to X(T^2)$ we have a planar algebraic curve $\mathcal{A}(K) = C^{-1}im(\iota^*)$. The ideal of planar algebraic curve is principle. A monic generator of this ideal is the **A-polynomial**. It is a Laurent polynomial in two variabls l and m. Using Culler and Shalens mechanism for relating points at infinity of the character variety of a three-manifold group with incompressible surfaces, a great deal of information about the geometry and topology of the knot complement is carried by the A-polynomial.

Placing a collar on the boundary of M_K there is an inclusion map

$$\iota : K_{-1}(T^2) \to K_{-1}(M_k). \tag{99}$$

Applying $C^* : K_{-1}(T^2) \to \mathbb{C}[l^{\pm 1}, m^{\pm 1}]$ to the kernel of ι and extending to get an ideal and then taking the radical recovers the A-ideal.

There is an obvious extension to skein algebras [18].

Define the B-ideal to be the kernel of

$$\iota : K_A(T^2) \to K_A(M_K). \tag{100}$$

It is no longer a two sided ideal. However, gluing the cylinder over the torus in so that the 0 end lies in the interior of the knot complement makes it a left ideal.

Next, map $ker(\iota)$ into $\mathcal{W}_A$ by C and extend to get a left ideal of $\mathcal{W}_A$. That is the **noncommutative A-ideal** is ${}^eC(ker(\iota)$ [18].

The algebra $\mathcal{W}_A$ is not a principle ideal domain. Instead start with rational functions in m and adjoin $l^{\pm 1}$ to get $\mathbb{C}(m)[l, l^{-1}]_A$ where we still require the noncommutation relation $lm = A^2 ml$. This is a principle ideal domain. Since $\mathcal{W}_A \leq \mathbb{C}(m)[l, l^{-1}]_A$ we can extend the left ideal to this domain to get a principle ideal. A monic generator of this ideal is the **noncommutative A-polynomial**.

If A is a root of unity, it is easy to see that the noncommutative A-polynomial is nontrivial. However, it is an open question whether it is always nontrivial. We found though that the noncommutative A-ideal annihilates the data from the Jones polynomial. When the noncommutative A-ideal is nontrivial we found that the colored Jones polynomials satisified a special kind of recursion formula derived from the action of the skein algebra of the torus on the skein module of a solid torus.

Lê and Garoufalidis [21] found a way around this, by instead formally defining a module over the exponentiated Weyl algebra, and proving the module is holonomic via an inductive process for proving that modules over the Weyl algebra are holonomic. In this case being holonomic reduces to having a nontrivial annhilator. The generator of the annihilator in the localization $\mathbb{C}(m)[l, l^{-1}]_A$ is their definition of the noncommutative A-polynomial. The AJ-conjecture states that the shape of the recursive formula for the colored Jones polynomials looks a lot like the A-polynomial. It has been proved true in many cases, mostly by proving that it coincides with our definition of the noncommutative A-polynomial.

What is missing is a coordinate free description of the localized Kauffman bracket skein module.

5. Lecture IIb

Consider the mapping class $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ The only irreducible representation of $K_\zeta(T^2)$ fixed by this mapping is the representation with $(a, b) = (1, 1)$.

To be clear, here is what the matrices look like in the case $n = 3$.

$$\rho(l) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \rho(m) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^{-2} & 0 \\ 0 & 0 & \zeta^{-4} \end{pmatrix} \tag{101}$$

Following the section on Skolem-Noether for mortals, we need to compute the action of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ on $E_{1,1}, \ldots, E_{1,n}$ and read off their first columns

Letting $e_{ij} = q^{-i,j} l^i m^j$. Recall that $E_{1,1} = \frac{1}{3}(e_{0,0} + e_{0,1} + e_{0,2})$. Since the matrix induced by $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is only well defined up to a scalar, we leave the $\frac{1}{3}$ by the wayside. Next, to write out $E_{i,1}$ we just multiply $E_{1,1}$ by $e_{i-1,0}$.

Hence

$$3E_{i,1} = e_{i,0}(e_{0,0} + e_{0,1} + e_{0,2}). \tag{102}$$

Next we apply the automorphism from $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

$$A_{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}}(e_{i-1,0}(e_{0,0} + e_{1,1} + e_{2,2})) = e_{2i-2,i-1}(e_{0,0} + e_{1,1} + e_{2,2}) \tag{103}$$

Next we evaluate this formula for $i \in \{1, 2, 3\}$ and read off the first columns.

$$(e_{0,0} + e_{1,1} + e_{2,2}) = \begin{pmatrix} 1 & 1/q^8 & 1/q^5 \\ 1/q & 1 & 1/q^{12} \\ 1/q^4 & 1/q^3 & 1 \end{pmatrix}. \tag{104}$$

Also,

$$e_{2,1} = \begin{pmatrix} 0 & 1/q^4 & 0 \\ 0 & 0 & 1 \\ 1/q^2 & 0 & 0 \end{pmatrix}, \tag{105}$$

and

$$e_{2,1}^2 = \begin{pmatrix} 0 & 0 & 1/q^4 \\ 1/q^2 0 & 0 & \\ 0 & 1 & 0 \end{pmatrix}. \tag{106}$$

Puting it all together, using the Skolem-Noether theorem, the quantum hyperbolic invariant of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is

$$C^{-1} = \begin{pmatrix} 1 & 1/q^2 & 1/q^5 \\ 1/q & 1 & 1 \\ 1/q^4 & 1/q^3 & 1 \end{pmatrix}. \tag{107}$$

.

Notice the action of $SL_2\mathbb{Z}$ as automorphisms of $M_n(\mathbb{C})$ gives rise, via the Skolem-Noether theorem a projective representation of the mapping class group of the torus. How does this relate to the Witten-Reshetikhin-Turaev representation.

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