GEAR PROBLEMS IN SKEIN THEORY

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This collection of problems is divided into four parts. The first part consists of exercises focused on deriving formulas in noncommutative algebra by induction. These would be a good starting place for graduate students. The second problem is open ended. Can you detect the geometry of a Sol manifold that is a mapping cylinder of an Anosov map of the torus from the asymptotics of the quantum hyperbolic invariants. The third problem set is to prove theorems about noncommutative tori that are in analogy with theorems that have been proved about the skein algebra. The fourth problem set are open problems in the theory of the Kauffman bracket skein algebra.

I want to thank Daniel Douglas for going through the exercises and figuring out how they worked, and helping me correct typos.

1. Formulas

These all work by induction. The first two are really elementary. The second two are useful formulas in the Kauffman bracket skein algebra, that are still pretty straightforward but require you to imagine diagrams.

1.1. **The quantum binomial theorem.** Let $q \in \mathbb{C} - \{0, 1\}$. The quantum integer l, denoted $[l]$ is defined as,

$$[l] = \frac{q^l - q^{-l}}{q - q^{-1}}.$$

The quantum factorial is defined recursively by $[0]! = 1$, and $[n]! = [n][n-1]!$. The quantum binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}.$$

Suppose that T is an associative algebra and $A, B \in T$ that satisfy $AB = q^2 BA$.

Exercise 1. *Prove that*

$$(A + B)^n = \sum_{k=0}^{n} q^{-k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix} A^k B^{n-k}.$$

1.2. **Chebyshev iteration.** Recall that the **Chebyshev polynomials of the first kind** $T_k(x)$ are defined by $T_0(x) = 2$, $T_1(x) = x$ and $T_k(x) = xT_{k-1}(x) - T_{k-2}(x)$. The **Chebyshev polynomials of the second kind** are defined by $s_0(x) = 1$, $s_1(x) = x$ and $s_k(x) = xs_{k-1}(x) - s_{k-2}(x)$.

Exercise 2.
- *Prove that for* $k \geq 2$, $T_k(x) = s_k(x) - s_{k-2}(x)$.
- *Prove that for all* $k, l \in \mathbb{N}$, $T_k(x)T_l(x) = T_{k+l}(x) + T_{|k-l|}(x)$.
- *Prove that if* $q \in \mathbb{C} - \{0\}$ *then* $T_k(q + q^{-1}) = q^k + q^{-k}$.
- *Prove that for all* $k, l \in \mathbb{N}$, $T_k(T_l(x)) = T_{kl}(x)$.

1.3. **Identities in the Kauffman bracket skein module.** We will work with *relative skeins*. A relative skein module also includes strips $[0,1] \times [0,1]$ that are embedded in M so that $[0,1] \times [0,1] \cap \partial M = \{0,1\} \times [0,1]$. Here we work with ambient isotopy relative to ∂M Each strip has a preferred side. Remember that in an oriented 3-manifold the orientation of a 2-dimensional submanifold is determined by a nonvanishing normal vector. Giving $[0,1] \times [0,1]$ the product orientation, the preferred side of the strip is the side the normal vector points to. We use the Kauffman bracket skein relations on relative skeins, making sure that in the ball where the skein relation takes place, the preferred side of all strips is up.

If $M = F \times [0,1]$ we can project all skeins to F. We assume that relative skeins have their boundaries in $\partial F \times [0,1]$ so that the boundary is a horizontal arc with it's preferred side up. That way we can represent skeins by diagrams with the blackboard framing in F, up to isotopy, the second and third Reidemeister moves and the move that locally flips monogons from one side to another without changing the local writhe. The simple diagrams are the ones with no crossings and no simple loops, and they form a basis for the relative skein module.

Let $Ann = S^1 \times [0,1]$. It is easy to see that $K_N(Ann)$ is isomorphic to $\mathbb{C}[x]$ where x is the framed link coming from the blackboard framing of the core of the annulus. Hence $1, x, x^2, \ldots$ is a basis for $K_N(Ann)$. A simple analysis of leading terms means that the $T_k(x)$ are also a basis. If $L \subset M$ is framed link, the result of replacing each component of L by $T_k(x)$ where x is the core of the particular annulus is called *threading* L with T_k. We denote it by $T_k(L)$.

The identity $T_k(q+q^{-1}) = q^k+q^{-k}$ has an analog in the relative skein of the annulus. The skein s_k is called the k-th spiral. It is the result of giving the blackboard framing to the diagram consisting of a single arc, having boundary $\{-1\} \times \{0,1\}$ that spirals around the annulus k-times. If k is positive it spirals counterclockwise and if k is negative

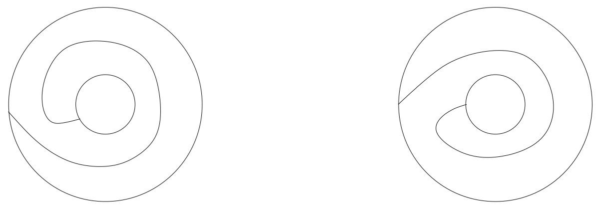


FIGURE 1. Two spirals

then it spirals clockwise. It intersects the arc $\{\mathbf{i}\} \times [0,1]$ transversely, and without bigons in k points. In Figure 1.3 we show s_1 and s_{-1}.

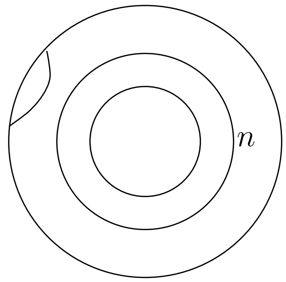
Exercise 3. *Prove* $T_k(x) * s_0 = q^k s_k + q^{-k} s_{-k}$, *and* $s_0 * T_k(x) = q^{-k} s_k + q^k s_{-k}$.

Let $S^1 \times [0,1] \times [0,1]$ be the solid torus viewed as a cylinder over the annulus. Choose two arcs in the vertical boundary and consider the relative Kauffman bracket skein module with respect to those arcs. We just draw the core of the diagrams involved, and assume that they are framed parallel to the sheet of paper. If we place an integer on a closed component then we mean that it is threaded with the corresponding Chebyshev polynomial of the second kind.

Up to isotopy there are exactly two relative skeins with no closed components and the blackboard framing, we call them ϵ_0 and ϵ_1. The skein ϵ_0 is the one that goes the short way around.

The skein module is an algebra under stacking. We let 1 denote the empty skein and x denote the core of the annulus with the blackboard framing. As a vector over $\mathbb{C}$ it is spanned by $1, x, x^2, \ldots$. However we could also thread the core with the Chebyshev polynomials of the second kind, $s_0(x) = 1, s_1(x) = x$ and $s_n(x) = x * s_{n-1}(x) - s_{n-2}$ and still have a basis.

The relative skein module is a module over the absolute skein module, and as such it is the direct sum of two cyclic modules generated by ϵ_0 and ϵ_1. To avoid many diagrams we use this fact to symbolically represent elements of the relative skein module. For instance $\epsilon_0 * s_n(x)$ is



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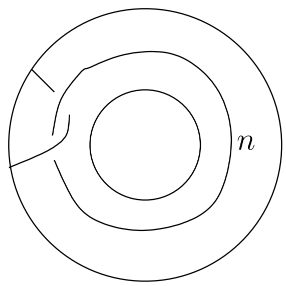
The goal of the exercise is to expand the skein c_n shown below



in terms of our basis of the module and algebra.

Exercise 4.

$$c_n = \sum_{i=0}^{n-1} q^{-2}(q^{2i+2} - q^{-2i-2})\epsilon_{n-i} s_i(x) + q^{2n}\epsilon_o s_n(x).$$

Here, we interpret the subscript of ϵ_{n-i} to be a residue class $\mod 2$. The proof may require that you derive the analogous identity involving ϵ_1.

2. Invariants of Anosov mapping classes of the torus

Recall that the torus is $T^2 = S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C}$ where we view S^1 as the unit circle. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z}$ then it defines an orientation preserving diffeomorphism of the torus by

$$T_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}(z,w) = (z^a w^c, z^b w^d). \tag{1}$$

Every orientation preserving self homeomorphism of $S^1 \times S^1$ is isotopic to a $T_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$. We say the mapping $T_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$ is **Anosov** if its eigenvalues are real and nonnegative. For instance, $T_{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}}$ is Anosov as when it is diagonalized it looks like,

$$\begin{pmatrix} \frac{1}{2}(3-\sqrt{5}) & 0 \\ 0 & \frac{1}{2}(3+\sqrt{5}) \end{pmatrix} \tag{2}$$

One of the degenerate geometries in dimension 3 is Sol. The underlying group is $\mathbb{R}^3$ where the multiplication is given by

$$(x,y,t)*(x',y',t') = (x+e^{-t}x', y+e^{t}y', t+t') \tag{3}$$

The mapping cylinder $M(\phi) = T^2 \times [0,1]/\phi$ of any Anosov map

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4}$$

of the torus carries Sol geometry. Here we are taking the quotient of $T^2 \times [0,1]$ by setting $(z,w,0)$ $(\phi(z,w),1)$. The fundamental group of the mapping cylinder is

$$\pi_1(M(\phi)) = < l, m, t | lm = ml, tl = l^a m^c t, tm = l^b m^d t > . \tag{5}$$

There is a natural picture of this acting on $\mathbb{R}^3$ where the longitude and meridian are translations whose direction lies in $\mathbb{R}^2 \times \{0\}$ and the action of ϕ is translation in the z-direction coupled by the action of ϕ in the $\mathbb{R}^2 \times \{0\}$ directions. By changing coordinates this can be made to look like a subgroup of Sol.

I don't think that there is a canonical volume that only depends on the topology of the underlying manifold, but if we require that the area of the torus cross sections are all 1, then I think that $\ln(\frac{1}{2}(3+\sqrt{5})$ is its volume as a Sol manifold.

It would be seriously cool, if this number could be detected by the quantum hyperbolic invariants of the mapping class. The growth of quantum invariants is very sensitive to which nth root of unity you choose.

Exercise 5. *Recall from Lecture IIb this induces an automorphism* $Q_{\phi,\zeta} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$. *The automorphism is induced by conjugation by a matrix* $C_{\phi,\zeta}$. *The matrix* $C_{\phi,\zeta}$ *is only determined up to a scalar multiple. One approach is to choose* $C_{\phi,\zeta}$ *so it has determinant* 1. *This still means that* $C_{\phi,\zeta}$ *is ambiguous up to a root of unity. Look at* $|Tr(C_{\phi,\zeta})|$ *as* n *grows and the argument of* ζ *is essentially fixed, and see how the growth rate in* n *depends on* $Arg(\zeta)$. **It is clear that the asymptotics of quantum invariants are drastically different depending on where the argument of the roots of unity**

being used head in the limit. *Also look for plus or minus the logarithm of the largest eigenvalue of ϕ to be derivable from the growth rate.*

3. Structure of Noncommutative Tori

These exercises explore the basic structure of noncommutative tori, and their representation theory when the variable is a $2n$th root of unity where n is odd. The results are in analogy to theorems proved about the Kauffman bracket skein algebra, but they are much simpler.

Let I be a finite set. An antisymmetric pairing on I is a function,

$$\sigma : I \times I \to \mathbb{Z} \tag{6}$$

so that for all $i, j \in I$, $\sigma(i,j) = -\sigma(j,i)$. We say that σ is unimodular if the determinant of the matrix corresponding to σ is ± 1.

Given σ we define the noncommutative torus

$$\mathbb{T}(\sigma) \tag{7}$$

to be the quotient of the algebra of noncommutative polynomials with complex coefficients in the variables x_i, by the two-sided ideal generated by $x_i x_j - q^{2\sigma(i,j)} x_j x_i$.

Assume that $q \in \mathbb{C} - \{0\}$.

Exercise 6. *Order I. Prove that monomials written so that the variables ascend are a basis for $\mathbb{T}(\sigma)$ over $\mathbb{C}$.*

For the sake of simplicity we will assume that $I = \{1, 2, \ldots, l\}$, and has been ordered by inclusion in $\mathbb{N}$. We say $\prod_i x_i^{k_i} < \prod_j x_i^{m_i}$ if when the first index i_0 where the exponents disagree $k_{i_0} < m_{i_0}$. If $J = (m_1, \ldots, m_l) \in \mathbb{Z}^n$ then

$$x^J = \prod_i x_i^{m_i}. \tag{8}$$

Given a nonzero element t of $\mathbb{T}(\sigma)$ it can be written uniquely as

$$t = \sum z_J x_J \tag{9}$$

where the $z_j \in \mathbb{C} - \{0\}$. Let J_{max} be the largest monomial appearing with nonzero coefficient in the expression for t. Define the **lead term** of t to be

$$ld(t) = z_{J_{max}} x^{J_{max}}. \tag{10}$$

Exercise 7. *Given $t, s \in \mathbb{T}(\sigma)$,*

$$ld(ts) = ld(t) ld(s). \tag{11}$$

Prove that $\mathbb{T}(\sigma)$ has no zero divisors.

Now assume that q is a primitive $2n$th root of unity where n is odd. Let $Z_0(\mathbb{T}(\sigma))$ denote the subalgebra of $\mathbb{T}(\sigma)$ generated by monomials all of whose exponents are divisible by n.

Exercise 8. *Prove that $Z_0(\mathbb{T}(\sigma))$ is contained in the center of $\mathbb{T}(\sigma)$.*

Exercise 9. *Prove that $\mathbb{T}(\sigma)$ is a free module of rank n^l over $Z_0(\mathbb{T}(\sigma))$ with basis given by monomials whose exponents range from 0 to $n-1$*

Let $\theta_i : \mathbb{T}(\sigma) \to \mathbb{T}(\sigma)$ be the map that replaces the variable x_i by $q^2 x_i$. It is an algebra automorphism.

Exercise 10. *Prove that*

$$\Theta_i = \frac{1}{n}\sum_{k=0}^{n-1} \theta_i^k : \mathbb{T}(\sigma) \to \mathbb{T}(\sigma) \tag{12}$$

has its image contained in the linear span of all monomials so that the exponent of i is divisible by n. Also prove that if the ith exponent of a monomial is already divisible by n, the map Θ_i sends it to itself. The idea here is that $\sum_{k=0}^{n-1} q^{2k} = 0$.

Exercise 11. *Prove that*

$$Tr(t) = \Theta_1 \circ \dots \Theta_l \tag{13}$$

has image contained in $Z_0(\mathbb{T}(\sigma))$, and is $Z_0(\mathbb{T}(\sigma))$-linear.

Exercise 12. *Prove that if σ is unimodular, the pairing*

$$\beta : \mathbb{T}(\sigma) \otimes \mathbb{T}(\sigma) \to Z_0(\mathbb{T}(\sigma)) \tag{14}$$

given by $\beta(s \otimes t) = Tr(st)$. Is nondegenerate in the sense that given $t \neq 0 \in \mathbb{T}(\sigma)$ there exists $s \in \mathbb{T}(\sigma)$ so that $\beta(t \otimes s) \neq 0$.

Here is another construction of Tr. If $t \in \mathbb{T}(\sigma)$ define

$$L_t : \mathbb{T}(\sigma) \to \mathbb{T}(\sigma) \tag{15}$$

by $L_t(s) = ts$. Since $\mathbb{T}(\sigma)$ is a free module of rank n^l over $Z_0(\mathbb{T}(\sigma))$ the map can be represented as an $n^l \times n^l$- matrix with coefficients in $Z_0(\mathbb{T}(\sigma))$. This matrix has a conventional trace, $tr(L_t)$.

Exercise 13. *Prove that*

$$Tr(t) = \frac{1}{n^l} tr(L_t). \tag{16}$$

Conclude that the pairing β is symmetric.

4. Open Problems

Embedding the Skein algebra of a closed surface in a noncommutative torus In [3] we embedded the skein algebra of the closed torus in a noncommutative torus. In [2], they produce an embedding of the skein algebra of any punctured surface with negative Euler characteristic into a noncommutative torus. How about skein algebras of closed surfaces?

Natural Definition of the Skein algebra This was the conjecture at the end of lecture IIa. Is the localized skein algebra the commutant a projective representation of the mapping class group. This might have something to do with the Hitchin connection.

Closed formula for the product in a punctured torus This seems really hard. In Frohman and Gelca [3] we found a closed formula for the product in the skein algebra of a torus. The skein algebra of the punctured torus is closely related, you can see presentations in [1]. It seems there should be an extension of our formula for the closed torus to the punctured torus with an error term that can be computed.

Formality of the noncommutative A-polynomial In [4] we constructed a left ideal of the skein algebra of the torus corresponding to a knot and proved the ideal annihilated a vector corresponding to the colored Jones polynomials of a knot. We conjectured that the noncommutative A-ideal was formal in the sense that it was characterized by the fact that it annihilated the colored Jones polynomial. In [5] a formal setting for annihilating the colored Jones polynomial was constructed. They conjectured that when specialized at -1, the annihilator of a module over the Weyl algebra constructed from the colored Jones polynomials was closely related to the A-polynomial. This is a conjecture that their setting is related to our setting. A lot of progress has been made on proving this is true, however what is missing is a conceptual link between the module over the Weyl algebra that they constructed and the module over the skein algebra that comes naturally from embedding the cylinder over the torus into the knot complement as a collaring of the boundary.

The skein module of a connected sum of $S^1 \times S^2$'s Bonahon and Wong's threading map means that the skein module of an oriented three-manifold is a module over the universal $SL_2\mathbb{C}$-character ring of is fundamental group.The threading map is no longer necessarily an embedding. Understanding this module structure would go a long way towards understanding quantum hyperbolic geometry.

Conjecture 1. *Let ζ be a primitive $2n$th root of unity. You can define the Kauffman bracket of skein in $\#_k S^1 \times S^2$ in much the same way as* *you do it in S^3, and it is a complex number that is a Laurent polynomial in ζ. There is a submodule $\chi \leq K_\zeta(\#_k S^1 \times S^2)$ that is isomorphic to the ring of$SL_2\mathbb{C}$-characters of the fundamental group of $\#_k S^1 \times S^2$. The quotient $K_\zeta(\#_k S^1 \times S^2)/\chi$ is isomorphic to $\mathbb{C}$ there skeins act as their Kauffman bracket.*

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