Research Statement
Sam Ballas

My research program is dedicated to the study of *locally homogeneous geometric structures on manifolds*, with a particular interest in *projective structures*. The study of these types of geometric structures dates back to Klein's 1872 Erlangen program and remains a vibrant and active area of research. Broadly speaking, I am mainly interested in two complementary questions. First, given a geometry, what types of manifolds can be locally modeled on that geometry and second, given a fixed manifold, what sorts of geometry does it admit? In addition to their independent interest, the answers to these types of questions often have implications in other areas of mathematics, such as number theory, representation theory, and dynamics. Another facet of my research is to discover and explore these connections.

A main focus of my research is on *convex projective structures*, which are loosely speaking generalizations of complete hyperbolic structures that share many of their important and beautiful properties, but are in general much more flexible. The study of these structures dates back to the work of Kuiper, Benzécri, Koszul, and Kac-Vinberg in the '50s and '60s [25, 12, 24, 31]. The area was reinvigorated by Goldman [20] and Benoist [9, 8, 10, 11] in the '90s and 2000s and is currently an active subject of research (see for example [13, 27, 28, 17, 15, 16, 18, 14, 22, 3, 6, 1, 2]). One important observation of Benoist is that there are non-hyperbolic, closed 3-manifolds that admit convex projective structures. In [11] he constructs several examples and shows that any closed 3-manifold that admits an indecomposable [1] convex projective structure has only hyperbolic pieces in its JSJ decomposition. In recent work with Danciger and Lee [3] (see writing sample) we are able to construct infinitely many new examples of non-hyperbolic manifolds that admit indecomposable convex projective structures. Our techniques involve deforming the (cusped) complete hyperbolic structure on each JSJ piece so that they have "totally geodesic" boundary and then gluing together the pieces along their boundary. Our methods also suggest a program for proving the converse to Benoist's result, namely that any closed 3-manifold with only hyperbolic pieces in its JSJ decomposition admits an idecomposable convex projective structure. We are currently in the process of executing this program which appears to require a diverse set of tools that include verified numerical computations and symplectic geometry.

A crucial result in hyperbolic geometry is *Thurston's Dehn filling theorem* which, roughly speaking, says that given a cusped hyperbolic 3-manifold, "most" Dehn fillings of M admit a hyperbolic structure. In recent work with Danciger, Lee, and Marquis [4], we are able to prove an analogue of Thurston's theorem in the context of convex projective geometry. Specifically, we are able to show that if M belongs to a certain (finite) collection of cusped 3-manifolds then infinitely many Dehn fillings of M admit a non-hyperbolic convex projective structure. The proof involves deforming cusped convex projective structures on M to certain "incomplete" projective structures and then showing that these incomplete structures can be completed to convex projective structures on Dehn fillings of M. Combining this result with work of Heusener–Porti [21] allows us to show that this new convex projective structure cannot be deformed to the hyperbolic structure on the Dehn filled manifold. These are the first examples of convex projective structures on a closed hyperbolic 3-manifold that are not deformations of the complete hyperbolic structure, answering a question originally posed by Benoist [7]. Furthermore, we conjecture that the collection of manifolds to which our result applies is much, in fact infinitely, larger and are currently working in this direction.

Another branch of my research has been focused on *Thin subgroups*, which are generalizations of lattices in semi-simple Lie groups. Specifically, a thin groups is an infinite index subgroup of a lattices that is also Zariski dense. These groups have attracted much interest in the last several years due to their interesting number theoretic properties and relative abundance of examples [23, 29, 19, 26, 30]. In recent work with D. Long [5] (see writing sample), we are able to use geometric techniques to produce the first known thin subgroups of $\mathrm{SL}(4,\mathbb{R})$ that are isomorphic to the fundamental group of a cusped hyperbolic 3-manifold. Our examples come from deforming the hyperbolic holonomy of the fundamental group of the figure-eight knot complement into $\mathrm{SL}(4,\mathbb{R})$ as in [1]. We are then able to use the geometry associated to the groups in order to certify that the resulting representations are both faithful and have Zariski dense image. We are currently working to extend our techniques to produce new examples of thin subgroups contained in both uniform and non-uniform lattices of other Lie groups, such as $\mathrm{SL}(n,\mathbb{R})$.

[1] Indecomposability is a mild non-triviality condition

References

[1] Samuel Ballas. Finite volume properly convex deformations of the figure-eight knot. *Geom. Dedicata*, 178:49–73, 2015.
[2] Samuel Ballas. Constructing convex projective 3-manifolds with generalized cusps. *ArXiv e-prints*, May 2018.
[3] Samuel Ballas, Jeffrey Danciger, and Gye-Seon Lee. Convex projective structures on nonhyperbolic three-manifolds. *Geom. Topol.*, 22(3):1593–1646, 2018.
[4] Samuel Ballas, Jeffrey Danciger, Gye-Seon Lee, and Ludovic Marquis. Properly convex dehn filling. *In preparation*, 2018.
[5] Samuel Ballas and Darren Long. Constructing thin subgroups commensurable with the figure-eight knot group. *Algebr. Geom. Topol.*, 15(5):3011–3024, 2015.
[6] Samuel Ballas and Ludovic Marquis. Properly convex bending of hyperbolic manifolds. *ArXiv e-prints*, September 2016.
[7] Yves Benoist. Benoist divisible convex sets problems. GEAR NETWORK RETREAT: 2012.
[8] Yves Benoist. Convexes divisibles. II. *Duke Math. J.*, 120(1):97–120, 2003.
[9] Yves Benoist. Convexes divisibles. I. In *Algebraic groups and arithmetic*, pages 339–374. Tata Inst. Fund. Res., Mumbai, 2004.
[10] Yves Benoist. Convexes divisibles. III. *Ann. Sci. École Norm. Sup. (4)*, 38(5):793–832, 2005.
[11] Yves Benoist. Convexes divisibles. IV. Structure du bord en dimension 3. *Invent. Math.*, 164(2):249–278, 2006.
[12] Jean-Paul Benzécri. Sur les variétés localement affines et localement projectives. *Bull. Soc. Math. France*, 88:229–332, 1960.
[13] Suhyoung Choi and William M. Goldman. Convex real projective structures on closed surfaces are closed. *Proc. Amer. Math. Soc.*, 118(2):657–661, 1993.
[14] Suhyoung Choi and Gye-Seon Lee. Projective deformations of weakly orderable hyperbolic Coxeter orbifolds. *Geom. Topol.*, 19(4):1777–1828, 2015.
[15] D. Cooper, D. D. Long, and M. B. Thistlethwaite. Computing varieties of representations of hyperbolic 3-manifolds into $\mathrm{SL}(4, \mathbb{R})$. *Experiment. Math.*, 15(3):291–305, 2006.
[16] D. Cooper, D. D. Long, and M. B. Thistlethwaite. Flexing closed hyperbolic manifolds. *Geom. Topol.*, 11:2413–2440, 2007.
[17] Daryl Cooper, Darren Long, and Stephan Tillmann. Deforming convex projective manifolds. *Geom. Topol.*, 22(3):1349–1404, 2018.
[18] V. V. Fock and A. B. Goncharov. Moduli spaces of convex projective structures on surfaces. *Adv. Math.*, 208(1):249–273, 2007.
[19] Elena Fuchs. The ubiquity of thin groups. In *Thin groups and superstrong approximation*, volume 61 of *Math. Sci. Res. Inst. Publ.*, pages 73–92. Cambridge Univ. Press, Cambridge, 2014.
[20] William M. Goldman. Convex real projective structures on compact surfaces. *J. Differential Geom.*, 31(3):791–845, 1990.
[21] Michael Heusener and Joan Porti. Infinitesimal projective rigidity under Dehn filling. *Geom. Topol.*, 15(4):2017–2071, 2011.
[22] Michael Kapovich. Convex projective structures on Gromov-Thurston manifolds. *Geom. Topol.*, 11:1777–1830, 2007.
[23] Alex Kontorovich. From Apollonius to Zaremba: local-global phenomena in thin orbits. *Bull. Amer. Math. Soc. (N.S.)*, 50(2):187–228, 2013.
[24] J.-L. Koszul. Déformations de connexions localement plates. *Ann. Inst. Fourier (Grenoble)*, 18(fasc. 1):103–114, 1968.
[25] N. H. Kuiper. On convex locally-projective spaces. In *Convegno Internazionale di Geometria Differenziale, Italia, 1953*, pages 200–213. Edizioni Cremonese, Roma, 1954.
[26] Jeffrey C. Lagarias, Colin L. Mallows, and Allan R. Wilks. Beyond the Descartes circle theorem. *Amer. Math. Monthly*, 109(4):338–361, 2002.
[27] Ludovic Marquis. Espace des modules marqués des surfaces projectives convexes de volume fini. *Geom. Topol.*, 14(4):2103–2149, 2010.
[28] Ludovic Marquis. Surface projective convexe de volume fini. *Ann. Inst. Fourier (Grenoble)*, 62(1):325–392, 2012.
[29] Hee Oh. Harmonic analysis, ergodic theory and counting for thin groups. In *Thin groups and superstrong approximation*, volume 61 of *Math. Sci. Res. Inst. Publ.*, pages 179–210. Cambridge Univ. Press, Cambridge, 2014.
[30] Peter Sarnak. Notes on thin matrix groups. In *Thin groups and superstrong approximation*, volume 61 of *Math. Sci. Res. Inst. Publ.*, pages 343–362. Cambridge Univ. Press, Cambridge, 2014.
[31] È. B. Vinberg and V. G. Kac. Quasi-homogeneous cones. *Mat. Zametki*, 1:347–354, 1967.