Exotic properly convex manifolds via Dehn filling

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Motivation

Let $M$ be a closed hyperbolic manifold.
Let $\mathbb{H}(M)$ be the space of hyperbolic structures on $M$. 
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Let $\mathcal{H}(M)$ be the space of hyperbolic structures on $M$.

Theorem
$\mathcal{H}(M)$ is connected.
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proof sketch:

- $\dim(M) = 2$: Fenchel-Neilsen coordinates on Teichmüller space
- $\dim(M) > 2$: Mostow rigidity.
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- $\dim(M) > 2$: Mostow rigidity.

Motivating Question: What happens if we look at other geometries?
Projective Geometry

Let $\mathbb{RP}^n$ be the space of lines through the origin in $\mathbb{R}^{n+1}$. 
Projective Geometry

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G = \text{PGL}_{n+1}(\mathbb{R}) := \text{GL}_{n+1}(\mathbb{R}) / \mathbb{R}^\times \text{I}
\]

$\mathbb{RP}^n$ is a geometry with automorphism group $G$. 
Convex Projective Geometry

Let $\tilde{H}$ be a hyperplane in $\mathbb{R}^{n+1}$
Let $H = P(\tilde{H})$ be the corresponding projective hyperplane
Convex Projective Geometry

Let \( \tilde{H} \) be a hyperplane in \( \mathbb{R}^{n+1} \).
Let \( H = P(\tilde{H}) \) be the corresponding projective hyperplane.
\( A_H := \mathbb{RP}^n \setminus H \) is an affine patch.
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Convex Projective Geometry

Let $\tilde{H}$ be a hyperplane in $\mathbb{R}^{n+1}$
Let $H = P(\tilde{H})$ be the corresponding projective hyperplane
$A_H := \mathbb{R}P^n \setminus H$ is an \textit{affine patch} (i.e. $\mathbb{R}P^n = \mathbb{R}^n \sqcup \mathbb{R}P^{n-1}$)
$\Omega \subset \mathbb{R}P^n$ is \textit{properly convex} if $\overline{\Omega}$ is a convex subset of \textit{some} affine patch
Convex Projective Geometry

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\( \Omega \subset \mathbb{R}P^n \) is properly convex if \( \overline{\Omega} \) is a convex subset of some affine patch
Let \( \Omega \) be properly convex.
Define
\[
PGL(\Omega) = \{A \in G \mid A(\Omega) = \Omega\}
\]
Convex projective geometry
Some examples
Convex projective geometry

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Convex Projective Geometry

Some examples

- $\tilde{T} = \mathbb{R}^3_+$ (positive orthant)
- $T = P(\tilde{T})$
- $\text{PGL}(T) \cong \text{Diag}_3 \times S_3 \subset \text{PGL}_3(\mathbb{R})$
Convex Projective Geometry

Some Examples

- $L$ a Lorentzian form on $\mathbb{R}^{n+1}$
- $C = \{ v \in \mathbb{R}^{n+1} \mid L(v, v) < 0 \}$
- $\mathbb{H}^n = P(C)$ (Klein Model)
- $\text{PGL}(\mathbb{H}^n) \cong \text{PO}(L)$
Convex Projective Manifolds

Let $\Omega$ be properly convex
Let $\Gamma \subset \text{PGL}(\Omega)$ be discrete
Convex Projective Manifolds

Let $\Omega$ be properly convex
Let $\Gamma \subset \text{PGL}(\Omega)$ be discrete
$\Omega/\Gamma$ is a \textit{convex projective manifold}
Some Examples
Complete Hyperbolic Manifolds

- $\Omega \cong \mathbb{H}^n$
- $\Gamma \subset \text{PGL}(\mathbb{H}^n)$ discrete

The $\mathbb{H}^n/\Gamma$ is a complete hyperbolic manifold
Some Examples

Hex Torus

- $\Omega \cong T$
- $\Delta \cong \langle \gamma_1, \gamma_2 \rangle \subset \text{Diag}_3$

$T/\Delta$ is a \textit{hex torus}
Convex Projective Structures

Let $M$ be a compact manifold

A *convex projective structure* on $M$ is $(f, \Omega/\Gamma)$

- $\Omega/\Gamma$ properly convex
- $f : M \to \Omega/\Gamma$ a diffeomorphism
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There is an equivalence relation generated by

- Isotopy of $f$
- Replace $\Omega/\Gamma$ with $\Omega'/\Gamma'$ where $\Omega' = A(\Omega)$, $\Gamma' = A\Gamma A^{-1}$ for $A \in G$
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Exotic Convex Projective Structures

Let $M$ be a closed hyperbolic manifold
Let $\text{CP}(M)$ be the set of equivalence classes
Topologize $\text{CP}(M)$ using $C^\infty$ topology on $C^\infty(\tilde{M}, \mathbb{RP}^n)$
Exotic Convex Projective Structures

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Let $\text{CP}(M)$ be the set of equivalence classes
Topologize $\text{CP}(M)$ using $C^\infty$ topology on $C^\infty(\tilde{M}, \mathbb{RP}^n)$

Definition
$p \in \text{CP}(M)$ is *exotic* if it is not the same connected component as $\mathbb{H}(M) \subset \text{CP}(M)$.

$p$ is exotic if it cannot be continuously deformed to a hyperbolic structure
Exotic Convex Projective Structures

Existence

When do exotic structures exist?
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- Dimension 2: No exotic structures (Goldman ’90)
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- Dimension \( \geq 4 \): ???

**Question**: Does every closed hyperbolic 3-manifold admit an exotic convex projective structure?
When do exotic structures exist?

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- Dimension 3: Infinitely many examples (B-Danciger-Lee-Marquis)
- Dimension $\geq 4$: ???

**Question**: Does every closed hyperbolic 3-manifold admit an exotic convex projective structure? (maybe yes!)
Some Tools

Let \([ (f, \Omega/\Gamma) ] \in \text{CP}(M) \).  
Define \( f_* : \pi_1 M \leftrightarrow \Gamma \subset G \) (holonomy)  
This is only well defined up to conjugacy in \( G \)
Some Tools

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Hol : CP($M$) → Rep($\pi_1 M$, $G$) := Hom($\pi_1 M$, $G$)/$G$

$[(f, \Omega/\Gamma)] \mapsto [f_*]$ (holonomy map)
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Theorem (Koszul)
\(\text{Hol}\) is an open map
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Moral: If you can deform the representation you can deform the structure.
Some Tools

- $M$ a closed hyperbolic 3-manifold
- $[(f_{hyp}, \mathbb{H}^n/\Gamma)] \in \text{CP}(M)$ the hyperbolic structure
- $\rho_{hyp} = (f_{hyp})_*$ hyperbolic holonomy
- $\mathfrak{g}$ the Lie algebra of $G$
- $H^1_{\rho_{hyp}}(\pi_1 M, \mathfrak{g})$ (twisted cohomology)
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Fact: Infinitesimally rigid \( \Rightarrow \) locally rigid \( \Rightarrow \) all non-hyperbolic structures are exotic.
Dehn Filling

Let $N$ be a manifold with $\partial N \cong T^2$.
Let $[\gamma] \in \pi_1(\partial N)$ be simple
Let $D$ be a solid torus with meridian $m$
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Let $[\gamma] \in \pi_1(\partial N)$ be simple
Let $D$ be a solid torus with meridian $m$
Let $N_\gamma$ be obtained by gluing $N$ and $D$ along boundaries by diffeomorphism mapping $\gamma$ to $m$ \textit{(Dehn filling of $N$ along $\gamma$)}
Dehn Filling

Let $N$ be the complement of the figure-8 knot
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**Theorem (Thurston’s Dehn Filling Theorem)**

All but finitely many Dehn fillings of $N$ admit a hyperbolic structure.
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**Theorem (B-Danciger-Lee-Marquis)**
Infinitely many Dehn fillings of $N$ admit exotic convex projective structures.
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Theorem (B-Danciger-Lee-Marquis)

Infinitely many Dehn fillings of $N$ admit exotic convex projective structures.

$N$ can be replaced by other 1-cusped hyperbolic manifolds.
Hyperbolic Dehn Filling

Let $\rho_{hyp} : \pi_1 N \to \text{PSL}(2, \mathbb{C})$ be the hyperbolic holonomy

Let $\Delta = \pi_1 \partial N = \langle \gamma_1, \gamma_2 \rangle \cong \mathbb{Z}^2$.

$\rho_{hyp}(\Delta) \subset G_p \cong \mathbb{R}^2$ \hspace{1cm} (stabilizer of $p \in \partial \mathbb{H}^3$)
Hyperbolic Dehn Filling

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$\rho_{hyp}(\Delta) \subset G_p \cong \mathbb{R}^2$  
(stabilizer of $p \in \partial \mathbb{H}^3$)
Hyperbolic Dehn Filling

Deform $\rho_{hyp}$ to non-conjugate $\rho' \in \text{Hom}(\pi_1 N, \text{PSL}(2, \mathbb{C}))$

$\rho'(\Delta) \subset G_\ell \cong \mathbb{C}^*$

(stabilizer of geodesic $\ell$)
Hyperbolic Dehn Filling

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$\rho'(\Delta) \subset G_\ell \cong \mathbb{C}^*$ (stabilizer of geodesic $\ell$)

$\rho'$ is the holonomy of an \textit{incomplete} hyperbolic structure on $N$. 
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$\rho' (\Delta) \subset G_\ell \cong \mathbb{C}^*$

($\text{stabilizer of geodesic } \ell$)

$\rho'$ is the holonomy of an incomplete hyperbolic structure on $N$.

Let $g_1 = \rho'(\gamma_1), g_2 = \rho'(\gamma_2)$

There are unique $(a, b) \in \mathbb{R}^2$ so that $\text{Dehn filling coordinates}$

$$a \log(g_1) + b \log(g_2) = 2\pi i$$
Hyperbolic Dehn filling
Dehn filling coordinates control geometry of the completion
Hyperbolic Dehn filling

Dehn filling coordinates control geometry of the completion

If \((a, b) \in \mathbb{Z}^2\) relatively prime
\[\delta = \gamma_1^a \gamma_2^b\] is simple curve in \(\ker \rho', \rho'(\Delta) \cong \mathbb{Z}\)
Hyperbolic Dehn filling

Dehn filling coordinates control geometry of the completion

If \((a, b) \in \mathbb{Z}^2\) relatively prime
\[\delta = \gamma_1^a \gamma_2^b\]
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The completion of incomplete structure is \(N_\delta\)
\(N_\delta\) has a hyperbolic structure!!
Hyperbolic Dehn Filling

Which $\delta$ arise from this construction?
Hyperbolic Dehn Filling

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Thurston: there is $k$ so that if

- $(a, b) \in \mathbb{Z}^2$
- $a, b$ relatively prime
- $a^2 + b^2 > k^2$

then $(a, b)$ are the Dehn filling coordinates of incomplete structure on $N$
Hyperbolic Dehn Filling

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Properly Convex Dehn Filling

Step 1

Deform $\rho_{hyp}$ to $\rho' \in \text{Hom}(\pi_1 N, G)$ where $\rho'$ is holonomy of convex projective structure with "generalized cusp" (Cooper-Long-Tillmann extension of Koszul Thm)
Properly Convex Dehn Filling

first deformation

$$\rho'(\Delta) \subset G^\Omega_\ell \cong \mathbb{R}_{dil} \oplus i\mathbb{R}_{uni} \cong \mathbb{C}$$

(stabilizer of \(\ell\) in \(\text{PGL}(\Omega)\))

There is (non-unique) \((a, b) \in \mathbb{R}^2\) so that

$$\rho'(\gamma_1^a \gamma_2^b) \in i\mathbb{R}_{uni}$$

\(a/b \in S^1 = \mathbb{R} \cup \{\infty\}\) is well defined (unipotent slope)
Properly Convex Dehn Filling

Step 2

Deform $\rho'$ to $\rho'' \in \text{Hom}(\pi_1 M, G)$ so that $\rho''(\Delta) \subset G^\rho''_\ell \cong \mathbb{C}^*$

(stabilizer of convex “nbhd” of $\ell$)
Properly Convex Dehn Filling

Step 2

Let $g_1 = \rho''(\gamma_1)$, $g_2 = \rho''(\gamma_2)$

Get *Dehn filling coordinates* $(a, b)$

$$a \log(g_1) + b \log(g_2) = 2\pi i$$
Properly Convex Dehn Filling

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Get \textit{Dehn filling coordinates} $(a, b)$

$$a \log(g_1) + b \log(g_2) = 2\pi i$$

Unipotent elements in $i\mathbb{R}_{uni} \subset G^\Omega_\ell$ deform to rotations in $G^\rho_\ell$ so $a/b$ is close to unipotent slope of $\rho'$
Properly Convex Dehn Filling

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\[\rho''(\Delta) \cong \mathbb{Z}\]

\[D \cong G_{\ell}^{\rho''} / \rho''(\Delta)\]  

(properly convex solid torus)
Properly Convex Dehn Filling

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(\textit{properly convex solid torus})

\[N_\delta\] admits a non-hyperbolic properly convex structure
Properly Convex Dehn Filling

Which $\delta$ arise
Properly Convex Dehn Filling

Which $\delta$ arise

$\text{Rep}(\pi_1, N, G)$

unipotent slope
Properly Convex Dehn Filling

Which $\delta$ arise

$\text{Rep} \left( \pi, N, G \right)$ unipotent slope $\text{IRU \{\infty\}}$

Exotic fillings
Properly Convex Dehn Filling

Which $\delta$ arise

$\text{Rep}(\pi, N, G)$

unipotent slope

$\mathbb{R} U \{\infty\}$

A positive proportion of fillings are exotic!
Properly Convex Dehn Fillings
Constructing the deformations

- \( \text{Rep}(\mathbb{Z}^2, G) \cong \mathbb{R}^6, \text{Rep}(\pi_1 N, G) \cong \mathbb{R}^3 \)
- There is a 3-dim locus of “pure” reps \( P \subset \text{Rep}(\mathbb{Z}^2, G) \) with repeated eigenvalue
- Contains holonomy of with generalized cusps and Dehn fillings
- Examine how \( P \) intersects \( \text{res} : \text{Rep}(\pi_1 N, G) \to \text{Rep}(\mathbb{Z}^2, G) \)
The Real Result

Theorem (B-Danciger-Lee-Marquis)

Let $M$ be a 1-cusped *infinitesimally rigid* 3-manifold with non-constant unipotent slope then a positive proportion of the Dehn fillings of $M$ admit exotic convex projective structures.
The Real Result

Theorem (B-Danciger-Lee-Marquis)

Let $M$ be a 1-cusped infinitesimally rigid 3-manifold with non-constant unipotent slope then a positive proportion of the Dehn fillings of $M$ admit exotic convex projective structures.

So far $M_{004}$ (fig-8), $M_{003}$ (fig-8 sister), $M_{007}$, and $M_{019}$ have been shown to satisfy these hypotheses.
Effective Questions

- Which cusped 3-manifolds are infinitesimally rigid?
- Which cusped 3-manifolds have non-constant unipotent slope?
- For a given $M$ what is the range of the unipotent slope map?
Thank you