Complex Projective Structures on Surfaces

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Overview

• Correspondences between an analytic object (ODEs & measured laminations) and geometric objects (complex projective structures)
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- **Today**: In certain cases we can make these correspondences are explicit
\( \mathbb{CP}^1 \) geometry

\( \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} \) \hspace{1cm} (Riemann Sphere)

\( \text{PSL}_2(\mathbb{C}) = \text{SL}_2(\mathbb{C})/\{\pm I\} \) \hspace{1cm} (Biholomorphisms of \( \mathbb{CP}^1 \))
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$\text{PSL}_2(\mathbb{C})$ acts on $\mathbb{CP}^1$ via linear fractional transformations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az + b}{cz + d}$$
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- There is no \( \text{PSL}_2(\mathbb{C}) \)-invariant metric on \( \mathbb{CP}^1 \)
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- There is no $\text{PSL}_2(\mathbb{C})$-invariant metric on $\mathbb{CP}^1$
- Circles are invariant and play the role of geodesics
Hyperbolic surfaces

Let $\Sigma := \Sigma_g$ be a surface of genus $g$ with $\chi(\Sigma) := 2 - 2g < 0$
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- $\mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \}$ (unit disk)
- $\mathbb{D}$ is a model of the **hyperbolic plane**
Hyperbolic surfaces

Let $\Sigma := \Sigma_g$ be a surface of genus $g$ with $\chi(\Sigma) := 2 - 2g < 0$

- $D = \{ z \in \mathbb{C} \mid |z| < 1 \}$ (*unit disk*)
- $D$ is a model of the *hyperbolic plane*
- $G_D := \text{Stab}_{\text{PSL}_2(\mathbb{C})}(D) = \text{Isom}(D) \neq \text{PSL}_2(\mathbb{R})$
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Theorem (Uniformization)

There is a discrete group $\Gamma \subset G_\mathbb{D}$ so that $\Sigma \cong \mathbb{D}/\Gamma$. 
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**Theorem (Uniformization)**

*There is a discrete group $\Gamma \subset G_{\mathbb{D}}$ so that $\Sigma \cong \mathbb{D}/\Gamma$.***

Let $T(\Sigma)$ be the space of hyperbolic structures on $\Sigma$

**Theorem**

*The space, $T(\Sigma) \cong \mathbb{R}^{6g-6}$*
Complex projective structures

Definition

Let $\Sigma$ be a surface. A *complex projective structure* on $\Sigma$ consists of charts from $\Sigma$ into $\mathbb{CP}^1$ whose transition functions are elements of $\text{PSL}_2(\mathbb{C})$.
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For $z \in U_1 \cap U_2$, $\phi_1(z) = g_{12} \phi_2(z)$
Development and holonomy

A more global approach

Using analytic continuation we can attempt to enlarge our charts
Development and holonomy

A more global approach

Using *analytic continuation* we can attempt to enlarge our charts

\[
\phi_1(U_1) \xrightarrow{g_{12} \circ \phi_2(U_2)} g_{12} \circ g_{23} \circ \phi_3(U_3) \\
\phi_2(U_2) \xrightarrow{g_{23} \circ \phi_3(U_3)} g_{12} \circ \phi_2(U_2) \\
\phi_3(U_3) \xrightarrow{g_{23}} g_{12} \circ \phi_2(U_2)
\]
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A more global approach

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\[ \phi_k(U_k) \quad \phi_k \]

\[ \phi_3(U_3) \quad g_23 \quad \phi_2(U_2) \quad \phi_2 \]

\[ \phi_1(U_1) \quad g_12 \circ \phi_2(U_2) \quad g_12 \circ g_23 \circ \phi_3(U_3) \quad \phi_1 \]

Not well defined on \( \Sigma \), We are really defining \( \text{dev}_r \Sigma \):

\[ D_{\mathbb{C}P^1}, \text{hol}_\pi \Sigma - \Gamma \cong \text{PSL}_2 \text{C} \]
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\text{dev} : \tilde{\Sigma} = \mathbb{D} \to \mathbb{CP}^1, \quad \text{hol} : \pi_1 \Sigma \cong \Gamma \to \text{PSL}_2(\mathbb{C})
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\[
[\ell] \mapsto g_{12} \cdots g_{m-1} m \phi_m(\ell(1))
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$$\text{hol} : \pi_1 \Sigma \cong \Gamma \to \text{PSL}_2(\mathbb{C})$$

$$[\gamma] \mapsto (g_{12} \cdots g_{k1})$$
Development and holonomy

Properties

• dev is called a developing map
• hol is called a holonomy representation
• dev is called a *developing map*
• hol is called a *holonomy representation*
• dev is a hol-equivariant local diffeomorphism
  i.e. $\text{dev}(\gamma \cdot z) = \text{hol}(\gamma) \cdot \text{dev}(z)$  \( \forall z \in \mathbb{D}, \gamma \in \pi_1 M \)
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- Constructing a complex projective structure is equivalent to constructing such an equivariant pair
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• Constructing a complex projective structure is equivalent to constructing such an equivariant pair

Let \( \mathcal{P}(\Sigma) \) be space of all complex projective structures on \( \Sigma \)
Let $\phi : \mathbb{D} \to \mathbb{C}$ be holomorphic and consider the differential equation

$$u'' + \frac{1}{2}\phi u = 0$$

(1)
Second order linear ODEs
Simply connected case

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**Theorem (Cauchy)**

*For any $c_1, c_2 \in \mathbb{C}$ there is unique $u : \mathbb{D} \to \mathbb{C}$ solution to (1) satisfying the initial condition $u(0) = c_1$ and $u'(0) = c_2$*
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The solutions to (1) form a 2-dimensional vector space
Second order linear ODEs

A local approach

Let \( U \subset \mathbb{C} \) be connected, and let \( \phi : U \rightarrow \mathbb{C} \) be holomorphic.

For \( p \in U \) there is a basis \( \{ u_1, u_2 \} \) of local solutions to (1).

Using analytic continuation we can attempt to extend \( u_1 \) and \( u_2 \) to all of \( U \).

Problem: when we analytically continue around a loop \( \gamma \) we may arrive at new solutions \( p v_1, v_2 \) which are not \( u_1, u_2 \).
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![Diagram showing connected regions $U_1$, $U_2$, and $U_3$ with a point $p$]
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**Problem:** when we analytically continue around a loop \( \gamma \) we may arrive at new solutions \( (v_1, v_2) \neq (u_1, u_2) \).
Second order linear ODEs
A global approach

Solution:

- There is $M(\gamma) \in \text{GL}_2(\mathbb{C})$ so that $M(\gamma) u_i = v_i$
- $M(\gamma)$ only depends on homotopy class of $\gamma$. 
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- Think of $u_i : \tilde{U} \rightarrow \mathbb{C}$ (defined on universal cover)
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• Think of $u_i : \tilde{U} \to \mathbb{C}$ \textit{(defined on universal cover)}
• For each $[\gamma] \in \pi_1(\Sigma) \cong \text{Deck}(\pi)$ and each $z \in \tilde{U},$

$$(u_i \circ [\gamma])(z) = M(\gamma)u_i(z)$$
Second order linear ODEs
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Get an equivariant pair:

$$ (u_1, u_2) : \tilde{U} \to \mathbb{C} \quad \quad \mathbf{M} : \pi_1(\Sigma) \to \text{GL}_2(\mathbb{C}) $$
An Example

Let $U = \mathbb{D}\setminus\{0\}$ and consider the equation

$$u'' + \frac{u}{4z^2} = 0$$

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$\phi : \mathbb{H} \rightarrow U, t \mapsto \exp(2\pi it)$ is a universal cover

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$$z^{1/2} = \exp \left( \log(z) / 2 \right) = \exp(\pi it)$$

$$\exp(\pi i(t + 1)) = \exp(\pi i) \exp(\pi it) = -\exp(\pi it) = -z^{-1/2}$$
Relation between constructions
Equations give structure

Let $\Sigma = \mathbb{D}/\Gamma$ be hyperbolic surface, $\phi : \Sigma \rightarrow \mathbb{C}$ holomorphic

- $u_1, u_2 : \mathbb{D} \rightarrow \mathbb{C}$ a basis of solutions to $u'' + 1/2u\phi = 0$

- $[M] : \pi_1(\Sigma) \rightarrow \text{PGL}_2(\mathbb{C})$ (projectivized) monodromy.
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$\text{dev} : \mathbb{D} \to \mathbb{CP}^1$, $z \xrightarrow{\text{dev}} \frac{u_1(z)}{u_2(z)}$

Let $[M(\gamma)] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
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$\text{dev} : \mathbb{D} \to \mathbb{CP}^1$, $z \mapsto \frac{u_1(z)}{u_2(z)}$ Let $[M(\gamma)] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$(\text{dev} \circ \gamma)(z) = \frac{(u_1 \circ \gamma)(z)}{(u_2 \circ \gamma)(z)} = \frac{au_1(z) + bu_2(z)}{cu_1(z) + du_2(z)}$$

$$= \frac{a \cdot \text{dev}(z) + b}{c \cdot \text{dev}(z) + d} = [M(\gamma)] \cdot \text{dev}(z)$$
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$$(\text{dev} \circ \gamma)(z) = \frac{(u_1 \circ \gamma)(z)}{(u_2 \circ \gamma)(z)} = \frac{au_1(z) + bu_2(z)}{cu_1(z) + du_2(z)} = a \cdot \text{dev}(z) + b$$

$$= \frac{c \cdot \text{dev}(z) + d}{c \cdot \text{dev}(z) + d} = [M(\gamma)] \cdot \text{dev}(z)$$

$(\text{dev}, [M])$ give a complex projective structure on $M$. 
Relations between the construction
Structure gives equations

If $f : \mathbb{D} \to \mathbb{C}$ is holomorphic the *Schwartzian* of $f$ is given by

$$S(f) = \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2$$
Relations between the construction
Structure gives equations

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- If \( u_1, u_2 \) solve \( u'' + \frac{1}{2} \phi u = 0 \) then \( S(u_1/u_2) = \phi \)
  
  \textit{(ODE “inverts” Schwartzian)}
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\begin{itemize}
  \item If \( u_1, u_2 \) solve \( u'' + \frac{1}{2} \phi u = 0 \) then \( S(u_1/u_2) = \phi \) (ODE “inverts” Schwartzian)
  \item \( (\text{dev}, \rho) \) a complex projective structure on \( \Sigma \) let \( \tilde{\phi} = S(\text{dev}) \)
\end{itemize}
Relations between the construction
Structure gives equations

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- If \( u_1, u_2 \) solve \( u'' + \frac{1}{2} \phi u = 0 \) then \( S(u_1/u_2) = \phi \)
  \((\text{ODE “inverts” Schwartzian})\)
- \((\text{dev}, \rho)\) a complex projective structure on \( \Sigma \) let \( \tilde{\phi} = S(\text{dev}) \)
- Equivariance of \( \text{dev} \Rightarrow \pi_1(\Sigma)\)-invariance of \( \tilde{\phi} \),
  get \( \phi : \Sigma \rightarrow \mathbb{C} \)
Relations between the construction
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If $f : \mathbb{D} \to \mathbb{C}$ is holomorphic the *Schwartzian* of $f$ is given by

$$S(f) = \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2$$

- If $u_1, u_2$ solve $u'' + \frac{1}{2} \phi u = 0$ then $S(u_1/u_2) = \phi$ *(ODE “inverts” Schwartzian)*
- $(\text{dev}, \rho)$ a complex projective structure on $\Sigma$ let $\tilde{\phi} = S(\text{dev})$
- Equivariance of dev $\Rightarrow \pi_1(\Sigma)$-invariance of $\tilde{\phi}$, get $\phi : \Sigma \to \mathbb{C}$
- Can form the ODE $u'' + \frac{1}{2} \phi u = 0$ on $\Sigma$
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- Equivariance of $\text{dev} \Rightarrow \pi_1(\Sigma)$-invariance of $\tilde{\phi}$, get $\phi : \Sigma \rightarrow \mathbb{C}$
- Can form the ODE $u'' + \frac{1}{2} \phi u = 0$ on $\Sigma$
  dev comes from a solution to this equation
Good News: Have constructions that relate an analytic object (ODEs) to a geometric object (complex projective structures)
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Bad News: The correspondence is opaque:
      Analytic properties $\leftrightarrow$ Geometric properties
Another Correspondence

Grafting

Let $\Sigma = \mathbb{D}/\Gamma$ hyperbolic, $\gamma \subset \Sigma$ a closed geodesic, $t \in \mathbb{R}^+$
Another Correspondence

Grafting

Let $\Sigma = \mathbb{D}/\Gamma$ hyperbolic, $\gamma \subset \Sigma$ a closed geodesic, $t \in \mathbb{R}^+$

We can produce a new complex projective structure, $\text{Gr}_{t\gamma}(X)$ on $\Sigma$ by \textit{grafting} in a Euclidean cylinder of height $t$

Figure: Picture from Dumas, \textit{Complex Projective Structures}
Another Correspondence

Grafting

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We can produce a new complex projective structure, $\text{Gr}_{t\gamma}(X)$ on $\Sigma$ by \textit{grafting} in a Euclidean cylinder of height $t$

Let $S$ be free homotopy class of s.c.c.'s. Get

$$\text{Gr} : S \times \mathbb{R}^+ \times \mathcal{T}(\Sigma) \rightarrow \mathcal{P}(\Sigma)$$
Thurston’s Theorem

Construction produces all complex projective structures
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Let $\mathcal{ML}(\Sigma)$ be measured laminations on $\Sigma$
(limits of weighted multicurves)
Thurston’s Theorem

Construction produces all complex projective structures

Let $\mathcal{ML}(\Sigma)$ be *measured laminations* on $\Sigma$

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Theorem (Thurston)

$$\text{Gr} : \mathcal{ML}(\Sigma) \times \mathcal{T}(\Sigma) \to \mathcal{P}(\Sigma)$$

is a homeomorphism.
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**Good News:** Every complex projective structure arises from grafting a hyperbolic surface.
Thurston’s Theorem

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*is a homeomorphism.*

**Good News:** Every complex projective structure arises from grafting a hyperbolic surface.

**Bad News:** The inverse procedure is fairly non-constructive.
A transparent case

Let $\Sigma = \Sigma_{0,3}$ (thrice punctured sphere)

Let $\sigma = (\text{dev}, \rho) \in \mathcal{P}(\Sigma)$

$\sigma$ is:

- tame if dev can be extended (meromorphically) to the punctures
- relatively elliptic if holonomy of peripheral curves is elliptic (conjugate to rotation $z \mapsto e^{i\theta}z$)
- non-degenerate if $\rho_p \pi_1 \Sigma_q$ has no finite orbits (e.g. no global fixed points)

Let $\mathcal{P}_d \Sigma_q$ be the space of tame, relatively elliptic, and non-degenerate structures on $\Sigma$
A transparent case

Let $\Sigma = \Sigma_{0,3}$ (thrice punctured sphere)
Let $\sigma = (\text{dev}, \rho) \in \mathcal{P}(\Sigma)$
$\sigma$ is:

- **tame** if dev can be extended (meromorphically) to the punctures
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Let $\mathcal{P}^\odot(\Sigma)$ be the space of tame, relatively elliptic, and non-degenerate structures on $\Sigma$
Examples
Triangular structures

Given a configuration of 3 circles in $\mathbb{CP}^1$ we can build (several) complex projective structures on $\Sigma$. (*triangular structures*)
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\[
\pi_1(\Sigma) \cong \langle \alpha, \beta \rangle, \\
\rho(\alpha) = R(C_2)R(C_3) \cong (z \mapsto e^{2i\theta}z), \\
\rho(\beta) = R(C_3)R(C_1) \cong (z \mapsto e^{2i\phi}z)
\]
Examples

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Examples
Triangular structures

Given a configuration of 3 circles in $\mathbb{C}P^1$ we can build (several) complex projective structures on $\Sigma$. *(triangular structures)*

The same circles support several different developing maps.
Grafting again

Given a triangular structure we can do 2 different types of grafting along embedded arcs.
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**Idea**: Slit open surface along an embedded arc and glue in copy of $\mathbb{CP}^1$
Grafting again

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Idea: Slit open surface along an embedded arc and glue in copy of $\mathbb{CP}^1$

- *Edge grafting* (blue)
- *Core grafting* (red)
Grafting again

Given a triangular structure we can do 2 different types of grafting along embedded arcs.

Idea: Slit open surface along an embedded arc and glue in copy of $\mathbb{C}P^1$

- **Edge grafting** (blue)
- **Core grafting** (red)

This grafting is discrete, not continuous!
Grafting Example

Edge grafting

How does grafting change the developing map?
Grafting Example

Edge grafting

How does grafting change the developing map?

\[ D \xrightarrow{\tau} \Delta \xrightarrow{\tau'} D \]

\[ \text{Gr} \]
Grafting Example

How does grafting change the developing map?

How does grafting change the holonomy?

*It doesn’t!!*
Theorem 1

Theorem 1 (B-Bowers-Casella-Ruffoni)

Let $\Sigma = \Sigma_{0,3}$ and let $\tau \in \mathcal{P}^\odot(\Sigma)$. Then $\tau$ is obtained from a triangular structure by a finite sequence of edge and core graftings.

The sequence of graftings and the triangular structure can be computed explicitly (Algorithmic).
Sketch of proof

- If \( \tau = (\text{dev}, \rho) \), then near each puncture dev looks like \( z \mapsto z^{\alpha/2\pi} \), for \( \alpha \in \mathbb{R} \) (*punctures have winding number*)
Sketch of proof

- If $\tau = (\text{dev}, \rho)$, then near each puncture dev looks like $z \mapsto z^{\alpha/2\pi}$, for $\alpha \in \mathbb{R}$ (punctures have winding number)
- Winding numbers determine $\tau \in \mathcal{P}^\circ(\Sigma)$ (Complex analysis)
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- Let $(2a, 2b, 2c)$ be winding numbers of $\tau$
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- Let $(2a, 2b, 2c)$ be winding numbers of $\tau$
- Edge grafting increases winding numbers by $(2\pi, 2\pi)$ and core grafting increases winding number by $4\pi$
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• Winding numbers determine $\tau \in \mathcal{P}^\circ(\Sigma)$ (Complex analysis)

• Let $(2a, 2b, 2c)$ be winding numbers of $\tau$

• Edge grafting increases winding numbers by $(2\pi, 2\pi)$ and core grafting increases winding number by $4\pi$

• If winding numbers are small there is a triangular structure with winding number $(2a, 2b, 2c)$ (angles are $a$, $b$, $c$)
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- If winding numbers are small there is a triangular structure with winding number $(2a, 2b, 2c)$ (angles are $a, b, c$)
- If some winding numbers are big can find, $a', b', c'$ small, and $k_a, k_b, k_c \in \mathbb{N}$, $(a', b', c') = (a, b, c) - \pi(k_a, k_b, k_c)$ so that there is a triangular structure with winding numbers $(2a', 2b', 2c')$ that can be grafted to $\tau$. 
Sketch of proof

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A typical example

Winding numbers are $2a = 9\pi$, $2b = 3\pi$, $2c = \pi$
A typical example

Winding numbers are \(2a = 9\pi, 2b = 3\pi, 2c = \pi\)

Then \(2a' = 3\pi, 2b' = \pi, 2c' = \pi, k_a = 3, k_b = 1, k_c = 0\)
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Complex analytic perspective

How do analytic properties of $u'' + 1/2 \phi u = 0$ correspond to geometric properties of complex projective structures?
Complex analytic perspective

How do analytic properties of \( u'' + \frac{1}{2} \phi u = 0 \) correspond to geometric properties of complex projective structures??

\[ \Sigma_{0,3} \cong \mathbb{CP}^1 \backslash \{0, 1, \infty\} \]

Theorem 2 (B-Bowers-Casella-Ruffoni)

\[ \tau \in \mathcal{P}^\circ (\Sigma_{0,3}) \text{ iff } \tau \text{ comes from a solution to } u'' + \frac{1}{2} \phi u = 0 \]

where \( \phi : \mathbb{CP}^1 \rightarrow \mathbb{C} \) is meromorphic with poles of order \( \leq 2 \) at \( \{0, 1, \infty\} \).
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**Theorem 2 (B-Bowers-Casella-Ruffoni)**

$\tau \in \mathcal{P}^\odot(\Sigma_{0,3})$ iff $\tau$ comes from a solution to $u'' + 1/2\phi u = 0$ where $\phi : \mathbb{CP}^1 \to \mathbb{C}$ is meromorphic with poles of order $\leq 2$ at $\{0, 1, \infty\}$.

*We can determine the winding numbers from the poles of $\phi$!!*
Determining winding number

- Near \( z = 0 \), \( \phi(z) = \frac{a}{z^2} + O(1/z) \)
Determining winding number

• Near $z = 0$, $\phi(z) = \frac{a}{z^2} + O(1/z)$
• Let $r_1, r_2$ solutions to $r(r - 1) + \frac{a}{2} = 0$
Determining winding number

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- Let $r_1, r_2$ solutions to $r(r - 1) + \frac{a}{2} = 0$
- Generically, solutions to $u'' + 1/2\phi u = 0$ are of form
  
  \[ u_1(z) = z^{r_1} h_1(z), \quad u_2 = z^{r_2} h_2(z) \]

  where $h_i(z)$ analytic and non-zero near $z = 0$.
  
  *(not quite if $r_1 - r_2 \in \mathbb{Z}$)*
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  \((not\ quite\ if\ r_1 - r_2 \in \mathbb{Z})\)
- $dev(z) = \frac{u_1(z)}{u_2(z)} = z^\theta M(z)$ where $\theta = r_1 - r_2$, $M(z)$ analytic and non-zero at $z = 0$
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*not quite if $r_1 - r_2 \in \mathbb{Z}$*

- $\text{dev}(z) = \frac{u_1(z)}{u_2(z)} = z^\theta M(z)$ where $\theta = r_1 - r_2$, $M(z)$ analytic and non-zero at $z = 0$
- $2\pi \theta$ is winding number and $\theta = \pm \sqrt{1 - 2a}$
Can we give specific relationship between geometric/analytic properties for general non-compact $\Sigma$?
Remaining questions

Can we give specific relationship between geometric/analytic properties for general non-compact $\Sigma$?

- Not an obvious candidate to replace triangular structures
Can we give specific relationship between geometric/analytic properties for general non-compact $\Sigma$?

- Not an obvious candidate to replace triangular structures
- Winding numbers don’t determine structure

(\textit{complex structure not unique})
Thank you!