Gluing Equations for Real Projective Structures on 3-manifolds

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Florida State University
(joint with A. Casella)

Trends in Low Dimensional Topology
June 23, 2020

Overview

1. Geometric Structures
   - What are they?
   - Projective structures
   - Why are they interesting?
   - How do we construct them?

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    - What are they?
    - Projective structures
    - Why are they interesting?
    - How do we construct them?
2. Gluing Equations
    - Tool for constructing projective structures
    - Philosophy: Gluing equations "discretize" the problem of constructing projective structures
    - Examples
        - Thurston's equations
        - B-Casella projective gluing equations

What is geometry?

Super accurate historical reenactment

Erlangen, Germany
(circa 1872)

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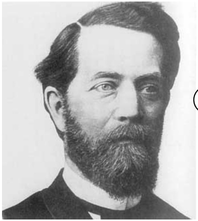

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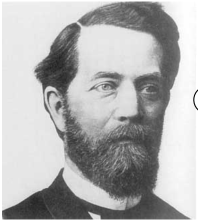

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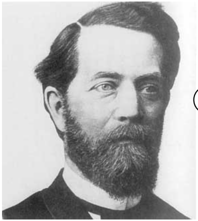

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Geometry is the study of properties of X that are invariant under G

Some examples

Type	Geometries	Geometric properties
Metric Geometries	$(\mathbb{S}^n, \mathrm{Isom}(\mathbb{S}^n))$ $(\mathbb{E}^n, \mathrm{Isom}(\mathbb{E}^n))$ $(\mathbb{H}^n, \mathrm{Isom}(\mathbb{H}^n))$	distance, angles, volume

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- $\mathbb{RP}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$
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Gives unified setting to study different geometries

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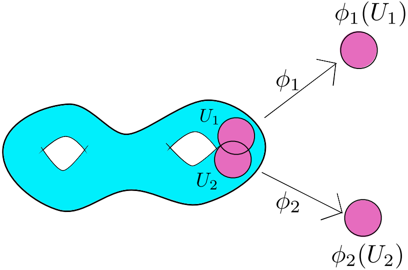

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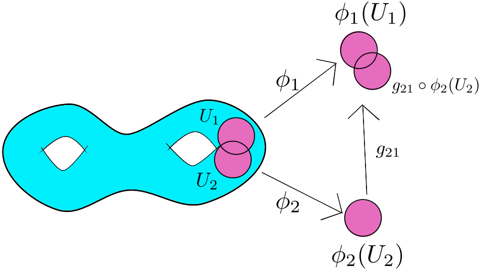

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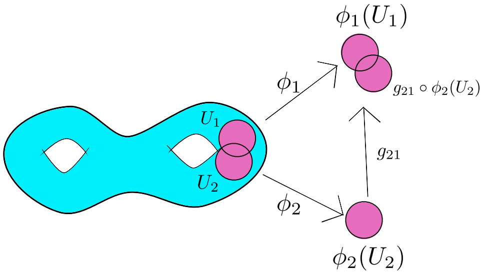
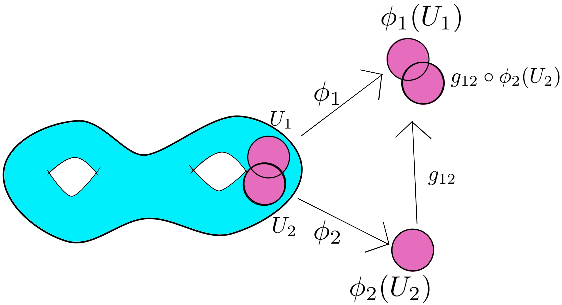


Analyticity ensures that transition functions are unique!

Development and holonomy

A more global approach



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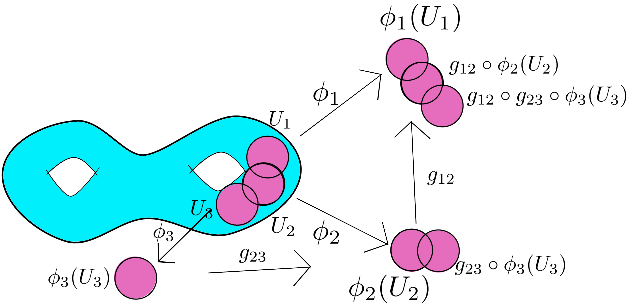

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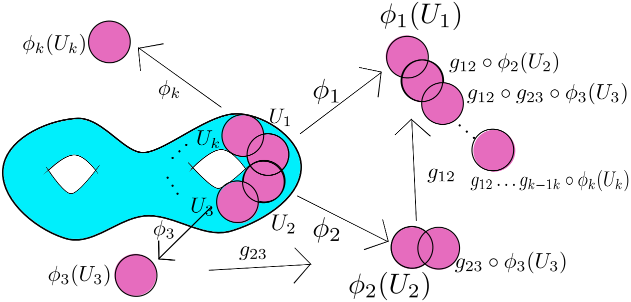

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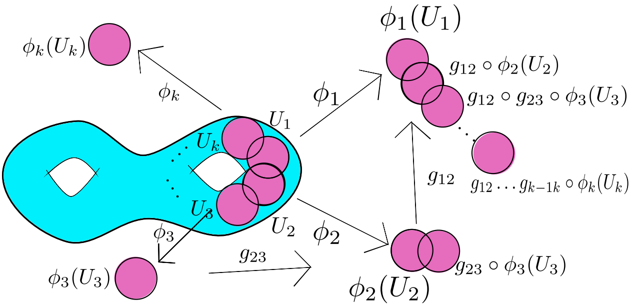

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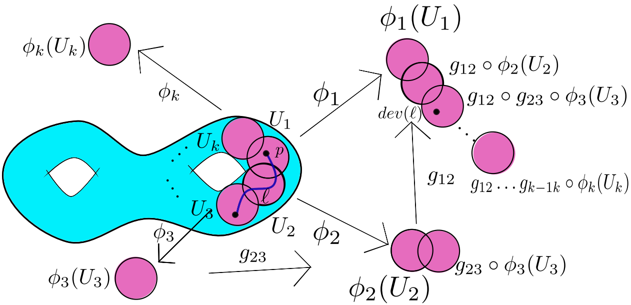

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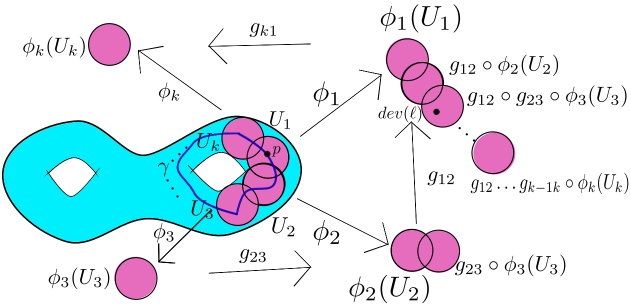

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Moral: To construct a projective structure you JUST need to find a representation $\rho : \pi_1 M \to \mathrm{PGL}_{n+1}$ and a ρ-equivariant local diffeomorphism $D : \widetilde{M} \to \mathbb{RP}^n$

Gluing equations

Ideal triangulations

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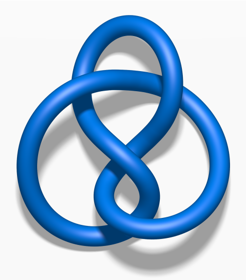
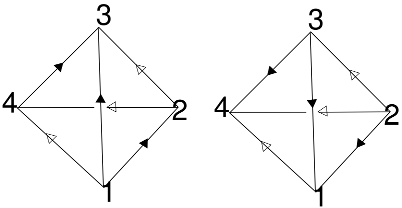
Ex: Figure-8 knot complement





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Variables and equations

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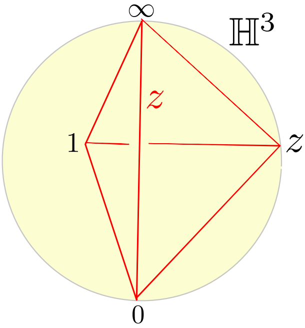
Equations

- Constraints must be imposed on shape parameters to ensure compatibility of charts
(existence of transition maps)
    i. Face equations: Ensure two tetrahedra can be glued together along a face
    ii. Edge equations: Ensure that the tetrahedra abutting an edge in $\mathcal{T}$ close up in $\mathbb{RP}^3$

Thurston's gluing equations

- Klein model gives $\mathbb{H}^3 \subset \mathbb{RP}^3$ and identifies $\partial\mathbb{H}^3$ with $\mathbb{C} \cup \infty$
- Fact: $\mathrm{Isom}^+(\mathbb{H}^3)$ acts simply triply transitively on $\partial\mathbb{H}^3$



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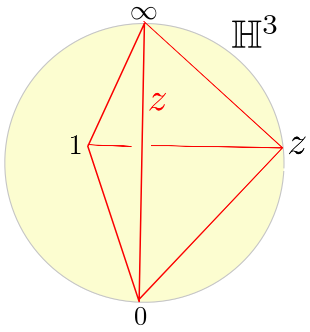

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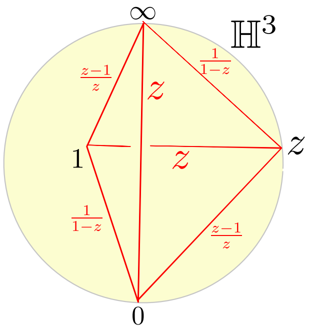

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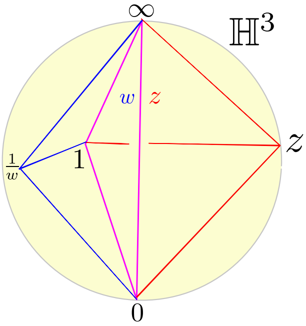
  - Ideal tetrahedra in $\mathbb{H}^3$ (modulo isometry) encoded by $z \in \mathbb{C}$, (*Thurston parameters*)
  - Given two ideal tetrahedra, there is a unique way to glue them along any face



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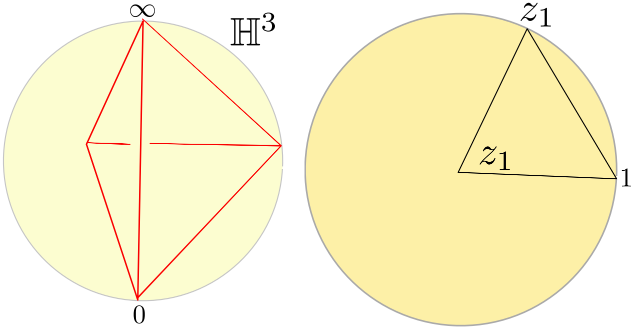

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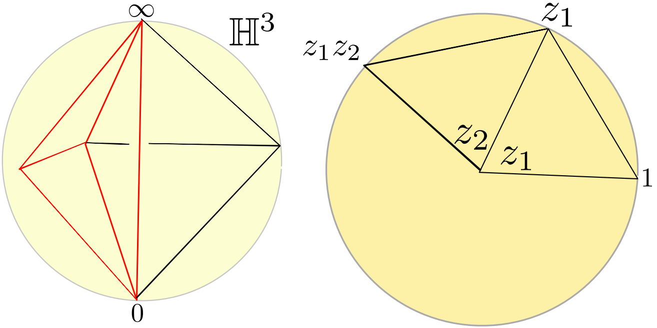

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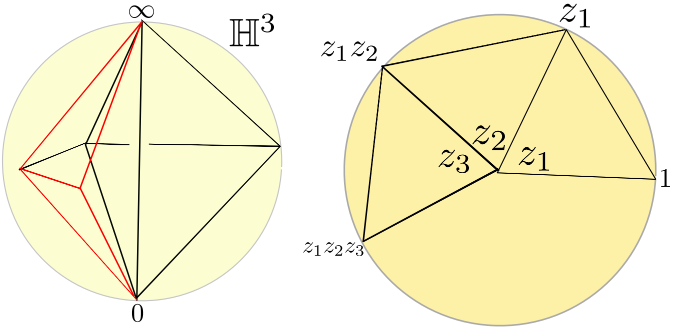

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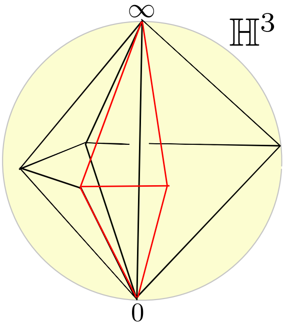

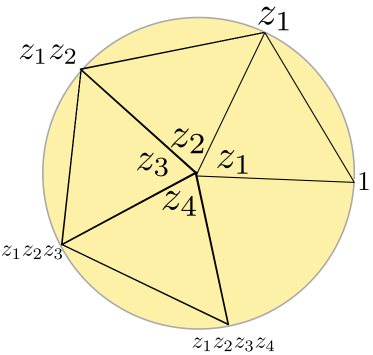

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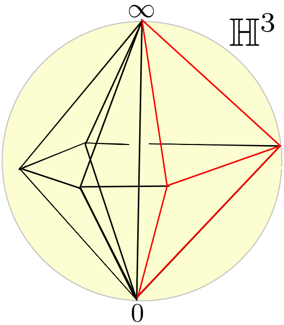

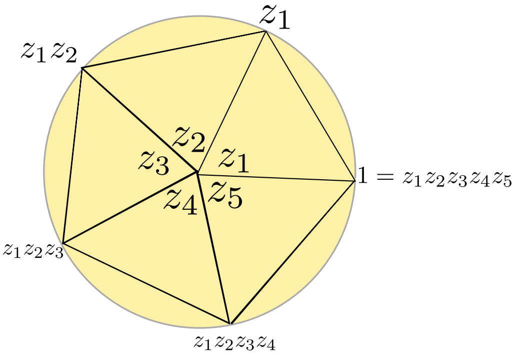

In order for the cycle to close up we need to impose an equation

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A solution to these equations is *geometric* if each component has positive imaginary part (No inside out tetrahedra)

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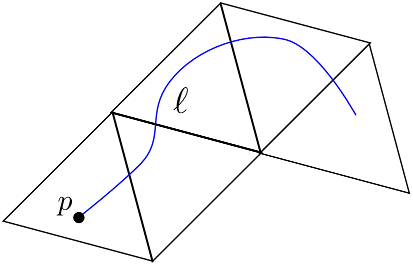
Each geometric solution to Thurston's gluing equations gives rise to a hyperbolic structure:

- A developing map dev : $\widetilde{M} \to \mathbb{H}^3$



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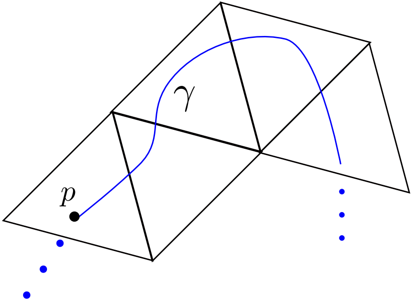

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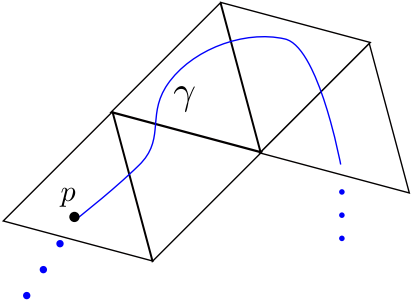

Gluing equations ensure this is well defined and equivariant!

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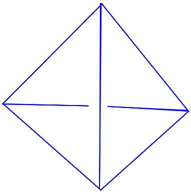

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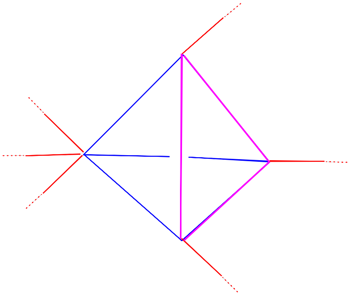
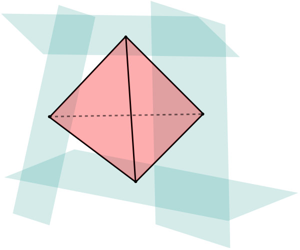
- Tetrahedra in $\mathbb{RP}^3$ are not uniquely determined by their vertices



Tetrahedra of flags

A *tetrahedron of (incomplete) flags in $\mathbb{RP}^3$* consists of

- 4 points in $\mathbb{RP}^3$
- A plane through each point
- There is a unique tetrahedron in $\mathbb{RP}^3$ whose vertices are the points and whose interior is disjoint from the planes



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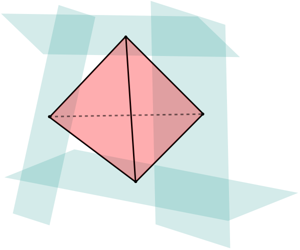
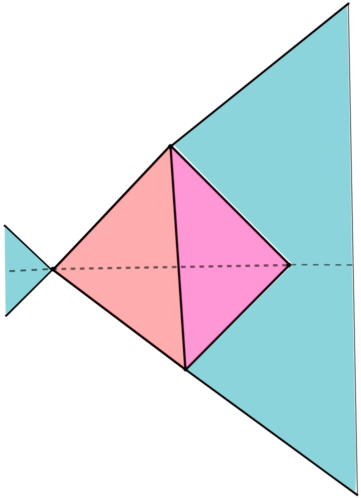


Don't have the previous problems for tet of flags

- Tetrahedron of flags has trivial PGL_4 stabilizer.
- A tetrahedron of flags determines a unique tetrahedron in $\mathbb{RP}^3$

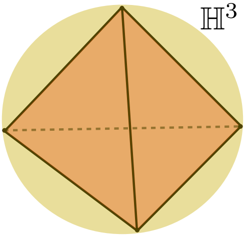
Why tetrahedra of flags?

Tetrahedra in $\mathbb{RP}^3$ are not determined by vertices



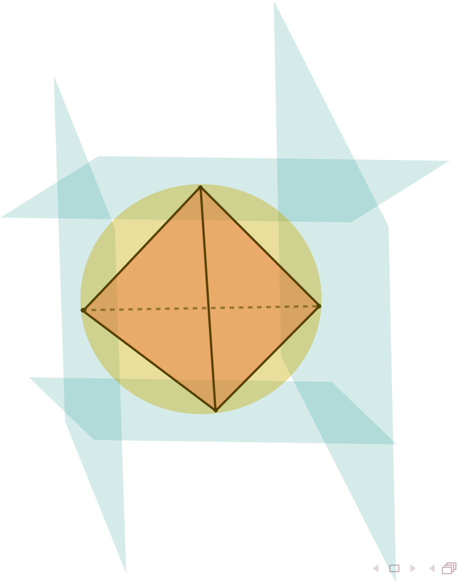
Why tetrahedra of flags?

$\mathbb{H}^3$ allows us to single out one of these tetrahedra



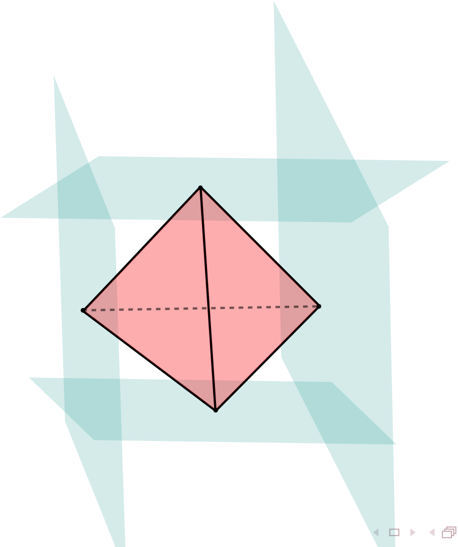
Why tetrahedra of flags?

An ideal tet in $\mathbb{H}^3$ determines a tet of flags via tangent planes



Why tetrahedra of flags?

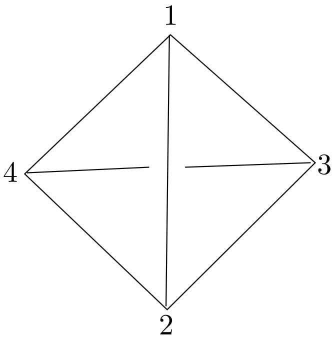
An ideal tet in $\mathbb{H}^3$ determines a tet of flags via tangent planes A tet of flags doesn't require planes to be tangent to $\mathbb{H}^3$



Projective gluing equations

Variables

Each tetrahedron of flags comes with coordinates



Projective gluing equations

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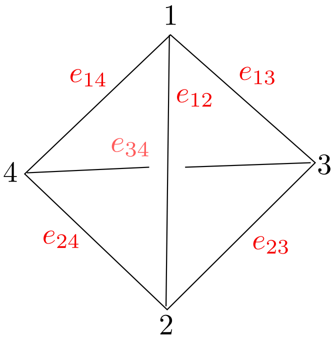
- 6 Edge coordinates: 1 per edge: Describe the shape of the tetrahedron of flags (not all independent!)

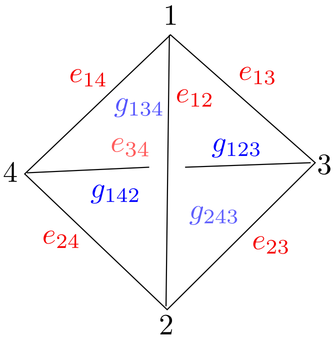


Projective gluing equations

Variables

Each tetrahedron of flags comes with coordinates

- 6 Edge coordinates: 1 per edge: Describe the shape of the tetrahedron of flags (not all independent!)
- 4 Gluing coordinates: 1 per face: Describe how adjacent tetrahedra of flags will be attached.



All coordinates are positive real numbers

Projective gluing equations

Equations

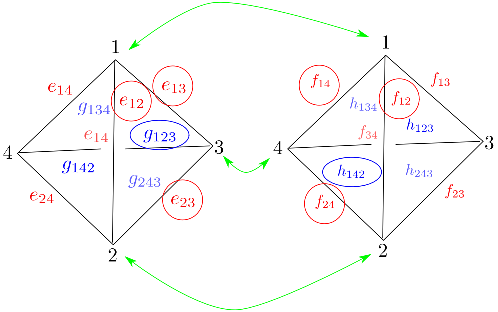
There are two types of equations

Projective gluing equations

Equations

There are two types of equations

- Face equations (2 per face)



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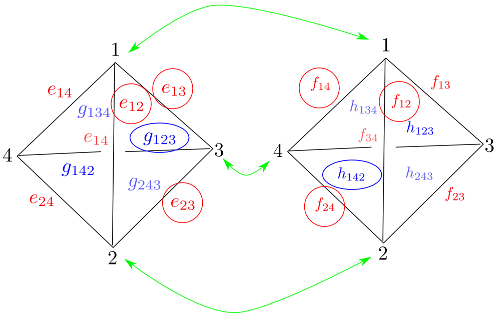


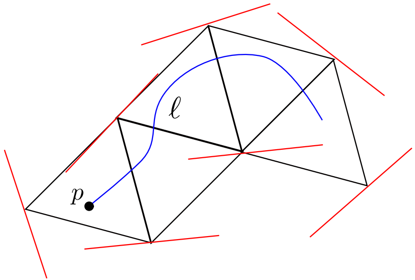
- Edge equations (5 per edge)
  Only involve variables "near edge"

Projective gluing equations

The projective structure

Each solution to the projective gluing equations gives rise to a projective structure:

- A developing map dev : $\widetilde{M} \to \mathbb{RP}^3$



Projective gluing equations

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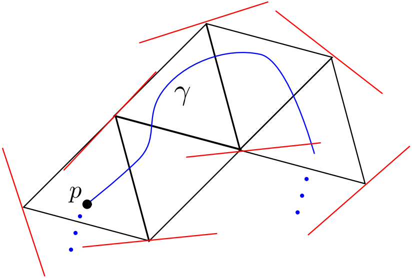

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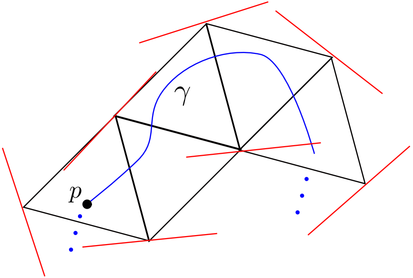
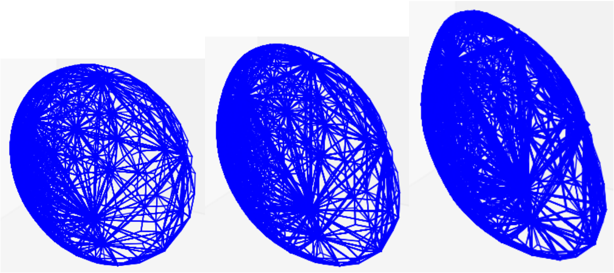


Gluing equations ensure this is well defined and equivariant!

Projective gluing equations

Nice properties

- Get lots of interesting projective structures: hyperbolic, anti de-Sitter, convex projective
- Geometric properties of structures manifest as algebraic properties of solutions.
- Numerically computing solutions can be automated (a la SnapPy)



Future directions

- Neumann-Zagier relationships
- Connections to quivers
- Geometric transitions
- Degenerations and tropicalizations

Thank you