Generalized cusps in convex projective manifolds

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(joint with D. Cooper and A. Leitner)

GEAR Retreat
Stanford University
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Outline

1. Cusps in finite volume hyperbolic manifolds
    - Geometry of cusps
    - Moduli space of cusps (a manifold)

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    - Generalize hyperbolic manifolds
    - Are more flexible
    - Occur as deformations of hyperbolic manifolds

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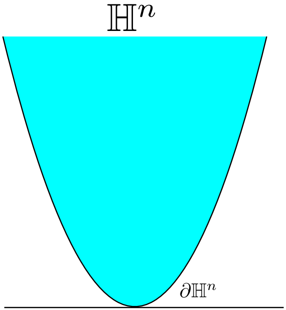
    - Geometry of cusps
    - Moduli space of cusps (a manifold)
2. Properly convex manifolds
    - Generalize hyperbolic manifolds
    - Are more flexible
    - Occur as deformations of hyperbolic manifolds
3. Generalized cusps
    - Occur as ends of properly convex manifolds
    - Have similar geometry to hyperbolic cusps
    - Have more complicated moduli space (stratified by orbifolds)
    - Exhibit interesting “transitional phenomena”

Hyperbolic space

Paraboloid model

Let $\mathbb{H}^n = \{(z, v) \in \underbrace{\mathbb{R}}_{\textit{Vertical}} \times \underbrace{\mathbb{R}^{n-1}}_{\textit{Horizontal}} \mid z > \frac{1}{2}|v|^2\} \subset \mathbb{R}^n \subset \mathbb{RP}^n$

- A projective model for hyperbolic space



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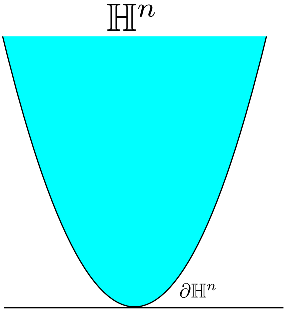

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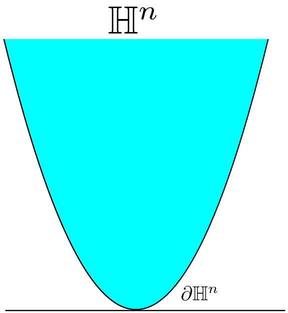
- Geodesics are (affine) straight lines



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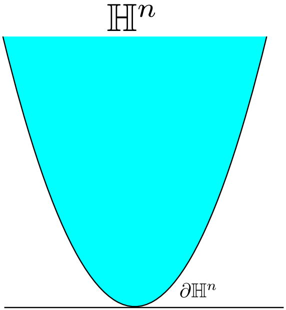

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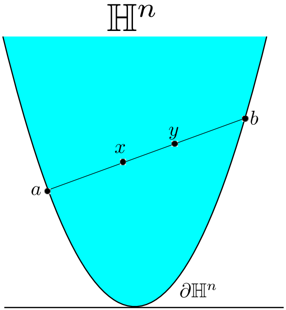
- Metric is given by $d_{\mathbb{H}^n}(x, y) = \frac{1}{2}\log([a : x : y : b])$



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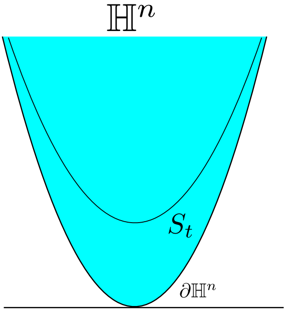
- Foliated by horospheres $S_t = \{(z,v) \in \mathbb{H}^n \mid z = \frac{1}{2}|v|^2 + t\}$, $t > 0$



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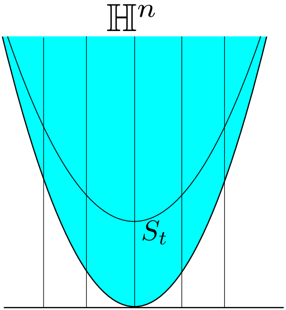

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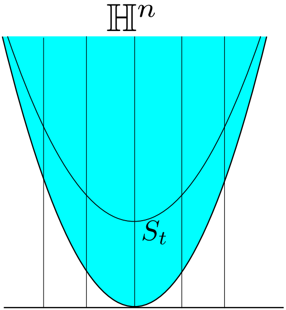
- The induced metric on S_t is flat and given by the Hessian of $z = \frac{1}{2}|v|^2$

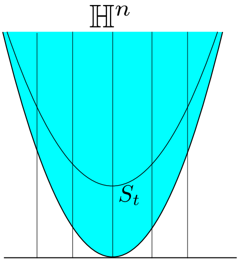


Cusps of hyperbolic manifolds

Paraboloid model

Consider the following subgroups of $\mathrm{Aff}_n(\mathbb{R})$

$$T = \left\{ \begin{pmatrix} 1 & u^t & \frac{1}{2}|u|^2 \\ 0 & I & u \\ 0 & 0 & 1 \end{pmatrix} \mid u \in \mathbb{R}^{n-1} \right\}, O = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid A \in O(n-1) \right\}$$



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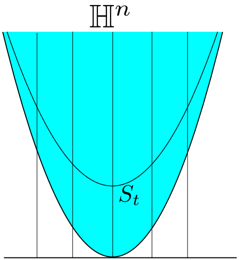

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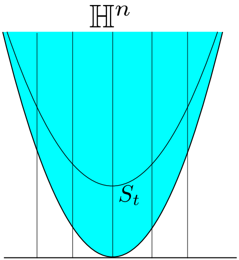
- $G := \langle T, O \rangle \cong T \rtimes O \cong \mathrm{Isom}(\mathbb{R}^{n-1})$



Cusps of hyperbolic orbifolds

Topology of cusps

Let $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ be a lattice and $M = \mathbb{H}^n/\Gamma$ be a complete hyperbolic n-orbifold.

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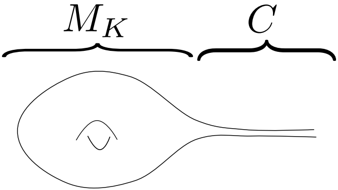
Using the “thick-thin” decomposition M can be decomposed into

$$M = M_K \bigsqcup_i C_i,$$

M_K compact and C_i finitely covered by $T^{n-1} \times [0, \infty)$.



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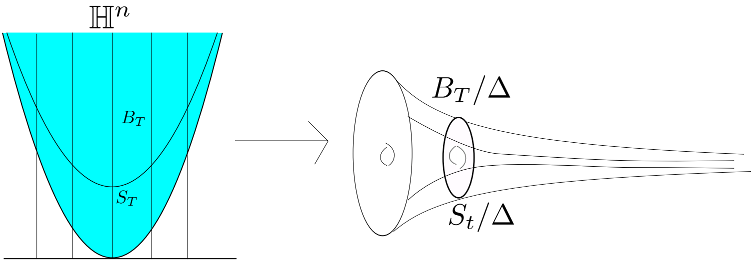

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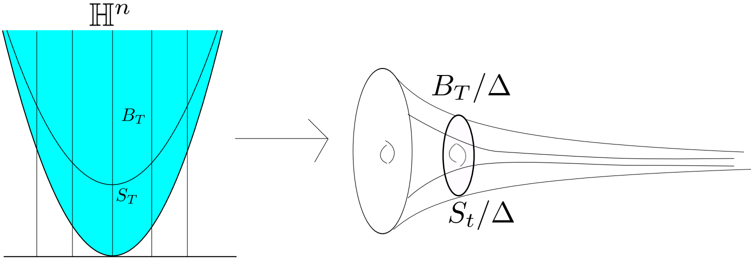
The S_t/Δ give a foliation of C by *Euclidean* $(n-1)$-orbifolds.



Cusps of hyperbolic manifolds

Moduli space of cusps

- A *marked torus cusp* is (f, C) where C is a cusp and $f : T^{n-1} \times [0, \infty) \to C$ is a diffeomorphism called a *marking*.

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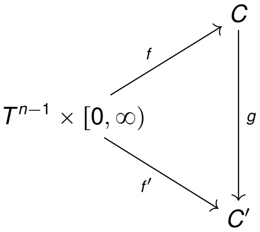

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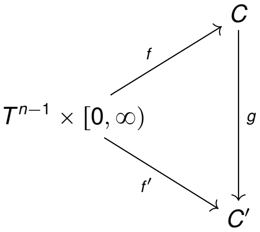

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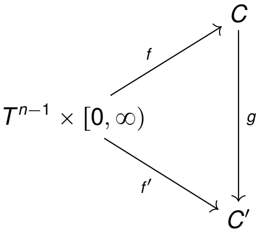

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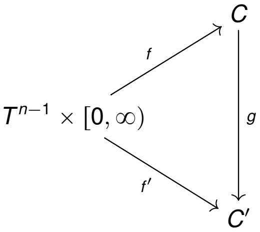

- Let $\mathfrak{T}$ be the space of equivalence classes of marked torus cusps
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- How can we use parameterize $\mathfrak{T}$?

Let $[(f, C)] \in \mathfrak{T}$

- Pick a basis for $T \cong \mathbb{R}^{n-1}$

Cusps of hyperbolic manifolds

Moduli space of cusps

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- $\mathfrak{T} \cong O(n-1)\backslash \mathrm{SL}^{\pm}_{n-1}(\mathbb{R})$

$\mathbb{RP}^n = \mathbb{R}^n \sqcup \mathbb{RP}^{n-1}$, so complement of any projective hyperplane is a copy of affine space called an *affine patch*.

Properly convex geometry

Properly convex domains

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$\Omega \subset \mathbb{RP}^n$ is *properly convex* if

1. $\overline{\Omega}$ is contained in an affine patch
2. Ω is a convex subset of an affine patch

Ω properly convex $\iff$ Ω is a bounded convex subset of some affine patch

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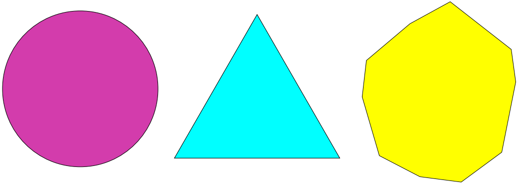
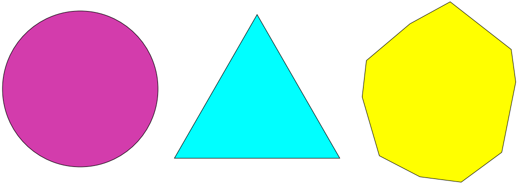


Ω determines a group $\mathrm{PGL}(\Omega) := \{A \in \mathrm{PGL}_{n+1}(\mathbb{R}) \mid A(\Omega) = \Omega\}$



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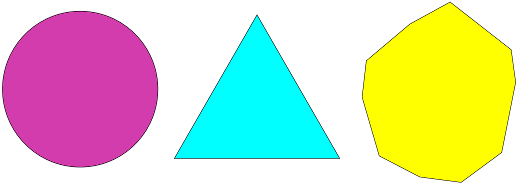

$\mathrm{PO}(2,1)$ $\quad$ $\mathrm{Diag}_3 \rtimes S_3$ $\quad$ $\{1\}$

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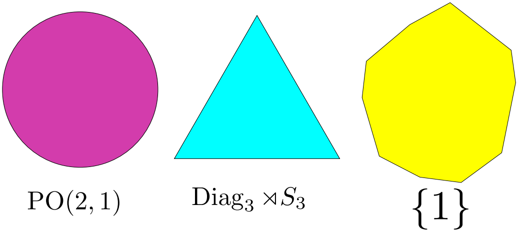

Generically, $\mathrm{PGL}(\Omega)$ is trivial

- Let Ω be properly convex and let $\Gamma \subset \mathrm{PGL}(\Omega)$ be discrete and torsion free.
- Ω/Γ is a *properly convex manifold*

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- Ω/Γ is a *properly convex manifold*
- Are there interesting properly convex manifolds?
  (Since $\mathrm{PGL}(\Omega)$ is generically trivial)
  Yes!

Properly convex manifolds

Example 1

A complete hyperbolic manifold $\mathbb{H}^n/\Gamma$ *is a properly convex manifold*

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Example 2

Deformations of properly hyperbolic manifolds

Properly convex manifolds

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Deformations of properly hyperbolic manifolds

Theorem 1 (Koszul)

If $M = \Omega/\Gamma$ *is a closed properly convex manifold and* $\Gamma' \leqslant \mathrm{PGL}_{n+1}(\mathbb{R})$ *is a small deformation of* Γ *then there is a properly convex domain* Ω' *such that* $\Gamma' \leqslant \mathrm{PGL}(\Omega')$ *is discrete and* $M \cong \Omega'/\Gamma'$

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Remark

Cooper–Long–Tillmann have proven a “relative version” of Koszul for *M* non-compact

Properly convex manifolds

Remark

By “bending” hyperbolic manifolds along totally geodesic hypersurfaces we get non-hyperbolic convex projective manifolds (Benoist, Marquis)

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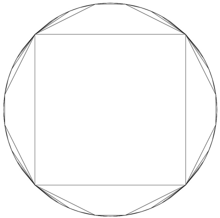

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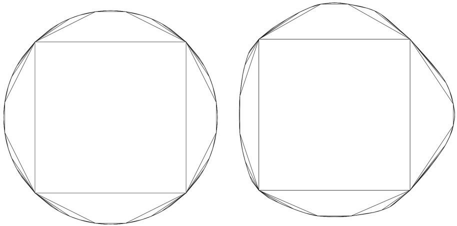

Generalized cusps

Motivation

- If $M = \mathbb{H}^n/\Gamma$ is a non-compact finite volume hyperbolic manifold
- Let $M' = \Omega/\Gamma'$ be a small properly convex deformation of M.
- What does the geometry of the ends of M' look like?

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  It's a *generalized cusp*

A properly convex manifold $C = \Omega'/\Delta$ is a *generalized cusp* if

- $C \cong \Sigma \times [0, \infty)$ with Σ compact
- Σ is a *strictly convex* hypersurface
  (lifts to Ω' are locally graphs of convex functions)
- Δ is vitually nilpotent (*or virtually Abelian*)

Generalized cusps

Questions

Let $C = \Omega/\Delta$ is a generalized cusp

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Let $C = \Omega/\Delta$ is a generalized cusp

1. What does Ω look like?
2. What does Δ look like?
3. What does the geometry of C look like?
4. What is the moduli space of generalized cusps?

Geometry of generalized cusps

Overview

Given an n-dimensional generalized cusp $C \cong \Omega'/\Delta$ we get

Geometry of generalized cusps

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- A properly convex domain $\Omega \subset \Omega'$ with smooth boundary (e.g. $B_T \subset \mathbb{H}^n$)

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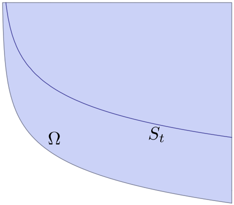
Examples

A quasi-hyperbolic cusp

Let $0 < \lambda_1 \leqslant \ldots \leqslant \lambda_{n-1}$

- Let $\Omega = \{(z, y) \in \underbrace{\mathbb{R}}_{\textit{vertical}} \times \underbrace{(\mathbb{R}_+)^{n-1}}_{\textit{horizontal}} \mid z > -\sum_i \lambda_i^{-1} \log(y_i)\}$



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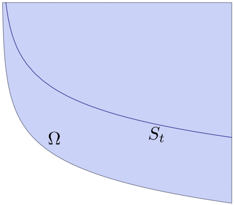

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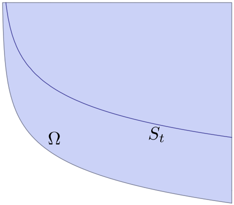

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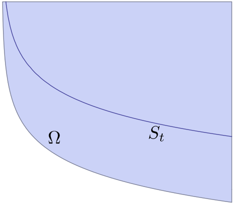

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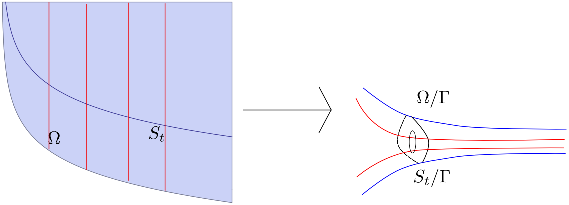

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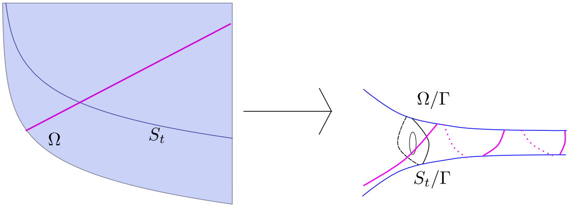

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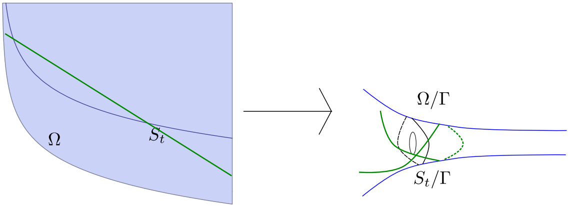

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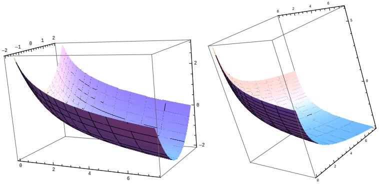

Figure: left: $\lambda_1 = 0, \lambda_2 = 1$. right: $\lambda_1 = \lambda_2 = 1$

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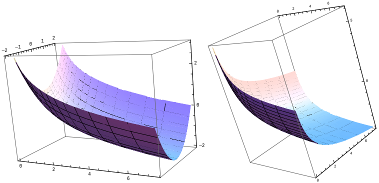

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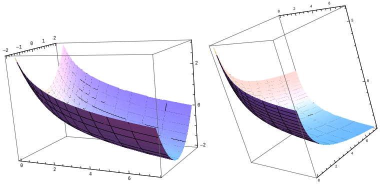

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Diagonalizable cusps

Let $0 < \lambda_0 \leqslant \ldots \leqslant \lambda_{n-1}$

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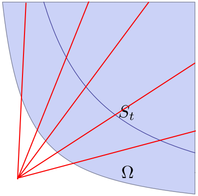

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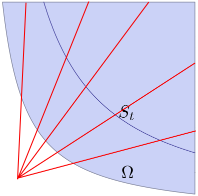

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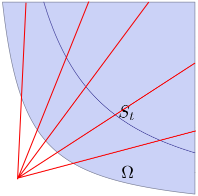

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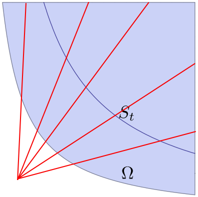

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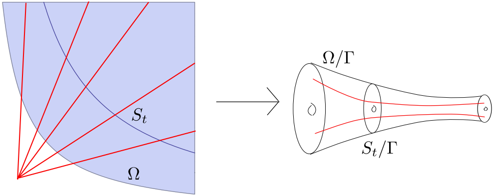

The big picture

Let $W_n = \{\lambda \in \mathbb{R}^n \mid 0 \leqslant \lambda_0 \leqslant \ldots \leqslant \lambda_{n-1}\}$ be the *Weyl chamber of* $\mathbb{R}^n$
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Remark 1

If $\exists c > 0$ such that $\lambda = c\lambda'$ then Ω_λ and $\Omega_{\lambda'}$ are projectively equivalent and G_λ and $G_{\lambda'}$ are conjugate.

Main Theorem

Theorem 2 (B–Cooper–Leitner)

Let $C = \Omega/\Gamma$ be an n-dimensional generalized cusp. Then there is a is a $\lambda \in W_n$, unique up to scaling, such that

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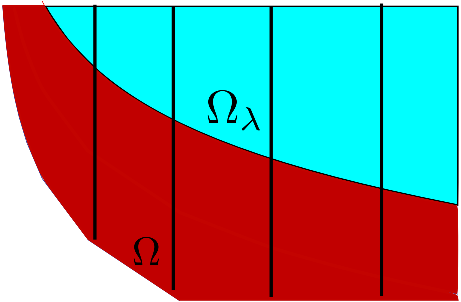

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- How can we use parameterize $\mathfrak{C}$?

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- Pick a basis for $T \cong \mathbb{R}^{n-1}$
- A *marked torus cusp* gives a *basis* for $\mathbb{R}^{n-1}$ (get $A \in \mathrm{GL}_{n-1}(\mathbb{R})$)
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Moduli space of generalized cusps

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Transitions

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Two perspectives

- Geometrically: Since cusp is non-compact, different parts look very different.
- Algebraically: Non-Hausdorff behavior of the character variety.

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Example

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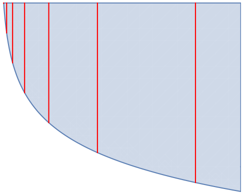

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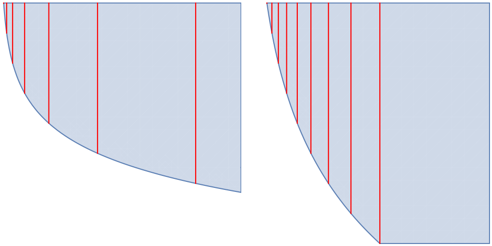

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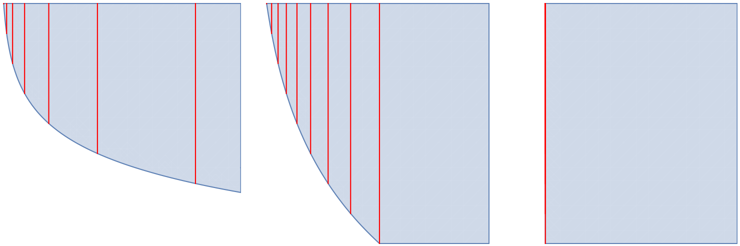

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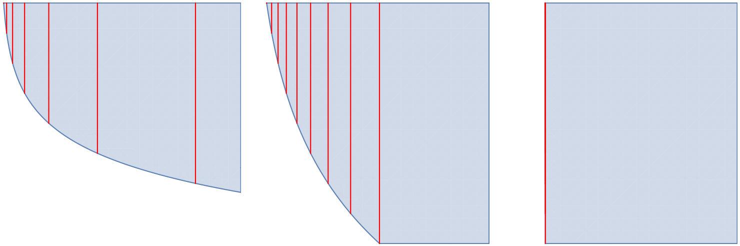

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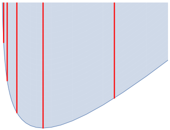

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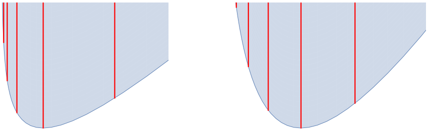

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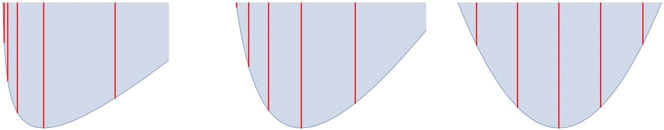

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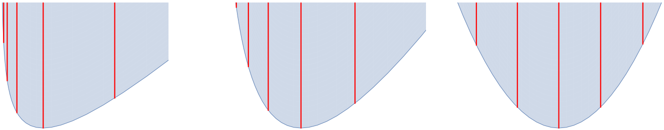
After applying a projective transform, $\Omega_{(0,b)}/\Gamma_b \to \Omega_{(0,0)}/\Gamma_0$



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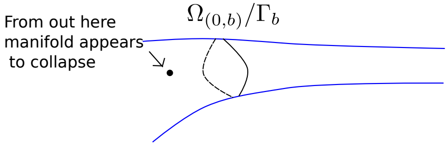

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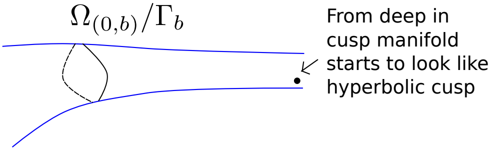

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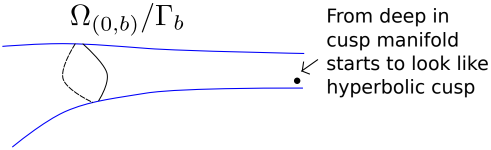

There are similar transitions anytime a coordinate in W_n goes to zero.

Transitions

Representation variety perspective

- Let $\mathcal{X}$ =Hom$(\mathbb{Z}^{n-1}, \mathrm{PGL}_{n+1}(\mathbb{R}))/\mathrm{PGL}_{n+1}(\mathbb{R})$ (character variety)

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  - $g_t \rho_t g_t^{-1} \to \rho'$ as $t \to 0$
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- Can we use the geometry of generalized cusps to give coordinates on the space of convex projective structures on a fixed manifold? (Fenchel-Nielsen coordinates)
- Better understand the action of the mapping class group on $\mathfrak{C}$ and study the quotient (unmarked cusps)

Thank you