A convex projective Dehn filling theorem

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Hyperbolic manifolds

Definitions

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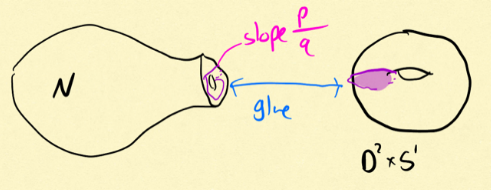
- M is *closed* if M is compact.
- "Most" closed 3-manifolds admit complete hyperbolic structures.

Thurston's hyperbolic Dehn filling theorem

Dehn filling

Let M be 3 manifold with $\partial N = T^2$. A *Dehn filling* of M is a *closed* manifold obtained by gluing $D^2 \times S^1$ to M along their boundaries.



Dehn fillings are parameterized by their *filling slope* $p/q \in \mathbb{Q} \cup \{\infty\}$ and are denoted by $M_{p/q}$

Thurston's theorem

Theorem 1

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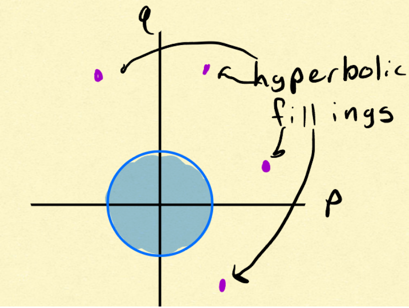

Convex projective manifolds

- A *properly convex projective manifold* is $M = \Omega/\Gamma$ where $\Omega \subset \mathbb{RP}^n$ is a *properly convex domain* in $\mathbb{RP}^n$ and $\Gamma \subset \mathrm{PGL}_{n+1}(\mathbb{R})$ is a discrete torsion free subgroup that preserves Ω.

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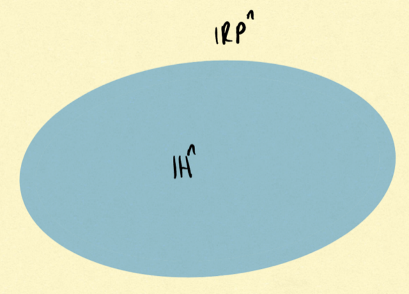
- M is *closed* if Ω/Γ is compact.
- Retain many "rank 1" features despite living in "high rank" Lie groups. ($\mathrm{SO}(n, 1)$ vs. $\mathrm{SL}_n(\mathbb{R})$, $n \geqslant 3$)

Convex projective manifolds

Examples

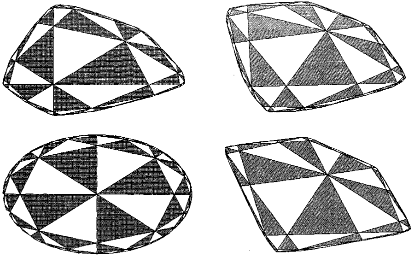
- $\mathbb{H}^n \subset \mathbb{RP}^n$ is properly convex, so complete hyperbolic manifolds are properly convex



Convex projective manifolds

Examples

- There are large deformation spaces of properly convex manifolds diffeomorphic to Σ_g, $g \geqslant 2$ (Goldman 1990) (à la Teichmuller space).



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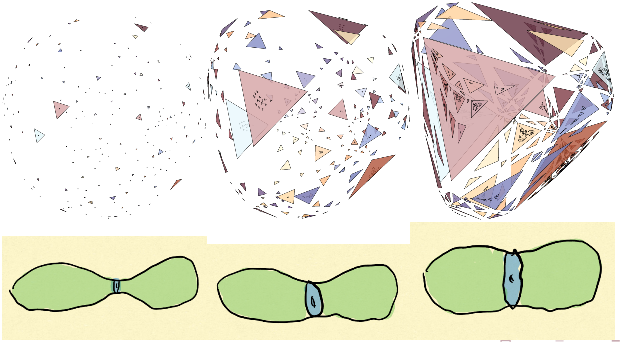

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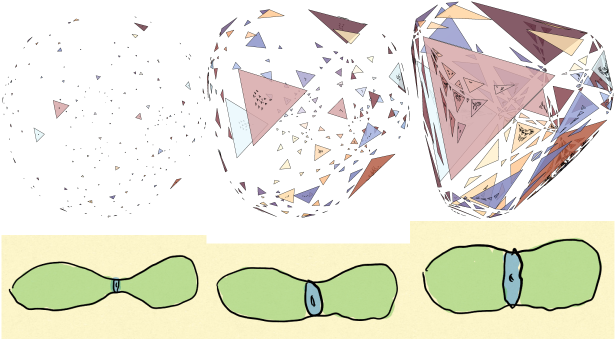
- Plus many, many others...



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rigidity

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Moral: It’s hard to construct non-hyperbolic properly convex manifolds diffeomorphic to close hyperbolic manifolds by deforming

Main theorem

Theorem 2 (B-Danciger-Lee-Marquis)

If $M \in \{m004$ (Figure-8), $m003$ (Figure-8 sister), $m007$, $m019\}$ then there is an interval, then there is $I \subset \mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$ so that

- *for all but finitely many filling slopes with $p/q \in I$ so that $M_{p/q}$ admits a non-hyperbolic properly convex projective structure.*

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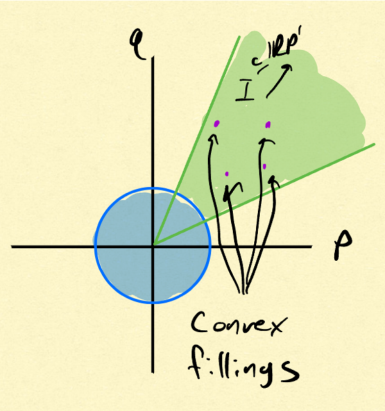
- Theorem actually provides verifiable hypotheses which above examples satisfy.
- Answer (negatively) a question asked by Benoist about connectedness of deformation spaces

Picture of main theorem



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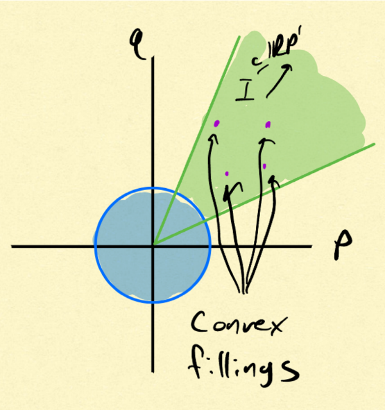

A "positive proportion" of all filling slopes have convex projective structures

Sketch of proof

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- Thurston showed all but finitely many a/b are realized

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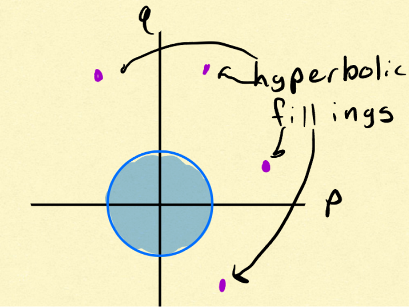

- ρ has Dehn filling coordinates ∞
- Complement of blue is neighborhood of ∞ in $\mathbb{R}^2 \cup \{\infty\}$

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- The unipotent slopes sweep out $I \subset \mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$ (I from the theorem).

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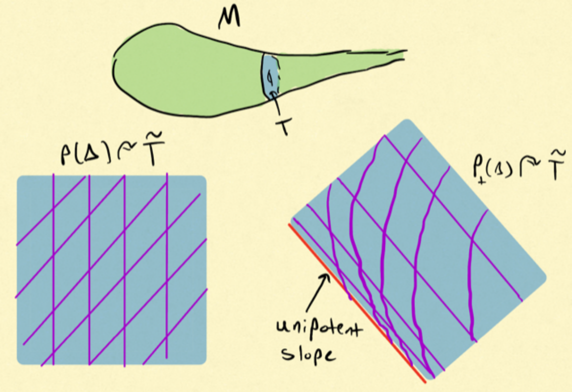

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- We show that all but finitely many relatively prime $(a_{t,s}, b_{t,s})$ in the "cone of I" can be realized

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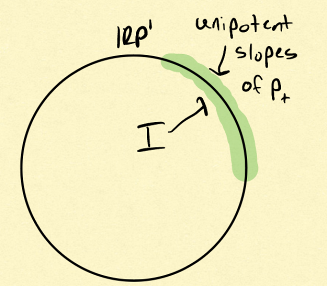

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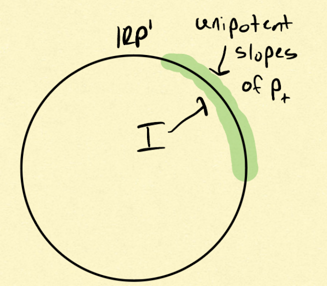

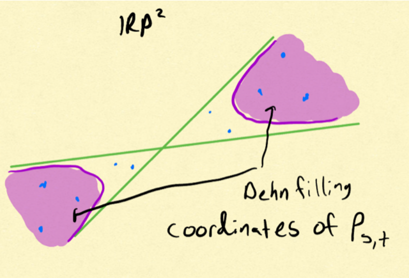

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  If yes we get a full analogue of Thurston's theorem

Thank you!