Algebra Qualifying Exam

January 7, 2023

Please value accuracy and precision: 7 approximate solutions will carry far less weight than 5 complete ones.

You may use standard results, provided you carefully state them ***in full****.*

Five problems carry **full credit**.

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(1) For any set S, let $\mathbb{Z}[S]$ denote the free abelian group over S. Now, let S, T be two sets, with S finite. Prove that

$$\mathbb{Z}[S \times T] \cong \bigoplus_{s \in S} \mathbb{Z}[T] .$$

(Hint: It is convenient to think of $S \times T$ as a disjoint union.)

(2) Let G be a group, and let H be a subgroup of G such that $[G : H]$ is finite. Prove that there exists a normal subgroup N of G such that $N \subseteq H$ and that $[G : N]$ is finite.

(3) Classify all groups of order 2023.

(4) Let k be a field, and $R = k[x]/(x^N)$ for some $N > 0$. Denote by $\bar{x}$ the class of x in R. (Recall that in a commutative ring R the localization at a prime $\mathfrak{p}$ means $S^{-1}R$, where $S = R \setminus \mathfrak{p}$.)

(i) Prove that $(\bar{x}) \subset R$ is prime.

(ii) Let M be an R-module such that $M_{(\bar{x})} = 0$. Prove that $M = 0$.

(5) Let $R = \mathbb{Z}[\sqrt{3}]$.

(i) Prove that $\mathbb{Z}[\sqrt{3}] \cong \mathbb{Z}[t]/(t^2 - 3)$.

(ii) Prove that $a \pm \sqrt{3}$, $a \in \mathbb{Z}$, is prime if and only if its norm $a^2 - 3$ is prime in $\mathbb{Z}$.

(iii) Let $f = x^n - r \in R[x]$. Assume $p = 13$ divides r, but p^2 does not. Prove that f is irreducible.

(6) Prove that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(n\mathbb{Z}), \mathbb{Z}/(m\mathbb{Z})) \cong \mathbb{Z}/(n, m)\mathbb{Z}$, where (n, m) denotes the gcd of n, m.

(7) Let $G = \mathrm{GL}_3(\mathbb{F}_2)$, the group of invertible 3×3 matrices with entries in $\mathbb{F}_2$, the field with two elements. Let G act on $M_3(\mathbb{F}_2)$, the set of 3×3 matrices, by $A \mapsto gAg^{-1}$, where $g \in G$ and $A \in M_3(\mathbb{F}_2)$. Classify the orbits of this action.