Algebra Qualifying Exam—Addendum

August 19, 2023

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(1) Let k be a field, and $R = k[x]/(x^N)$ for some $N > 0$. Denote by $\bar{x}$ the class of x in R. (Recall that in a commutative ring R the localization at a prime $\mathfrak{p}$ means $S^{-1}R$, where $S = R \setminus \mathfrak{p}$.)

(i) Prove that $(\bar{x}) \subset R$ is prime.

(ii) Let M be an R-module such that $M_{(\bar{x})} = 0$. Prove that $M = 0$.

(2) Let $R = \mathbb{Z}[\sqrt{3}] \cong \mathbb{Z}[t]/(t^2 - 3)$.

(i) Prove that $a \pm \sqrt{3}$, $a \in \mathbb{Z}$, is prime in R if and only if its norm $a^2 - 3$ is prime in $\mathbb{Z}$.

(ii) Let $f = x^n - r \in R[x]$ and assume that $f(x)$ is *re*ducible. If $4 + \sqrt{3}$ divides r, prove that 13 also divides r.

(3) Consider the ring $R = \mathbb{C}[x, y, z]/(xy - z^2)$. Is it an UFD?

(4) Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

over $k = \mathbb{F}_3$ (the field with three elements).

(i) Find the rational canonical form over k.

(ii) Is the Jordan form defined over k?

(5) Let k be a field, and consider a finite field extension $k \subseteq F$. Suppose $[F : k]$ is odd, and $\alpha \in F$ is such that $F = k(\alpha)$. Prove that $F = k(\alpha^2)$.