Algebra Qualifying Exam

August 20, 2022

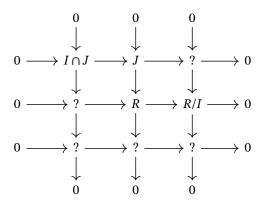
Please value accuracy and precision: 8 approximate solutions will carry less weight than 4 complete ones. You may use standard results, provided you carefully state them in full.

Six problems carry full credit.

- (1) Let $C = \operatorname{Mod}(R)$ be the category of R-modules, where R is a ring. Let $f : A \to M$ and $g : A \to N$ be two homomorphisms. Prove there exists an R-module, denoted $M \oplus_A N$, such that for any R-module P, and any pair of homomorphisms $u : M \to P$ and $v : N \to P$, such that $u \circ f = v \circ g$, there exists a unique homomorphism $M \oplus_A N \to P$. Also prove that such $M \oplus_A N$ is determined up to unique isomorphism.
- (2) Let G be a group with $|G| = 2 \cdot 3 \cdot 5^3 = 750$. Prove that G cannot be simple.
- (3) Let k be a field, and $R = k[x]/(x^N)$ for some N > 0. Denote by \bar{x} the class of x in R. (Recall that in a commutative ring R the localization at a prime $\mathfrak p$ means $S^{-1}R$, where $S = R \setminus \mathfrak p$.) Let M be an R-module such that $M_{(\bar x)} = 0$. Prove that M = 0.
- (4) Let $R = \mathbb{Z}[\sqrt{3}]$.
 - (i) Prove that $a \pm \sqrt{3}$, $a \in \mathbb{Z}$, is prime if and only if its norm $a^2 3$ is prime in \mathbb{Z} .
 - (ii) Let $f = x^n r \in R[x]$. Assume p = 13 divides r, but p^2 does not. Prove that f is irreducible.
- (5) Prove that the polynomial $f = x^3y^2 + x^2y^3 + x + y^3$ is irreducible in k[x, y], where k a field, assumed to be of characteristic zero, for simplicity.

Is the ideal I = (f) prime? Maximal?

(6) Let R be a commutative ring, and let I, J be two ideals. Fill the blanks in the following diagram in such a way that every row and column is short exact (justify your procedure):



- (7) Let I be the ideal (p, x) in $R = \mathbb{Z}[x]$, where p is a prime.
 - (a) Prove that I is not a free R-module.
 - (b) Provide a free resolution.
- (8) Let $G = \operatorname{GL}_3(\mathbb{F}_2)$, the group of invertible 3×3 matrices with entries in \mathbb{F}_2 , the field with two elements. Let G act on $M_3(\mathbb{F}_2)$, the set of 3×3 matrices, by $A \mapsto gAg^{-1}$, where $g \in G$ and $A \in M_3(\mathbb{F}_2)$. Classify the orbits of this action.