Algebra Qualifying Exam

August 20, 2022

Please value accuracy and precision: 8 approximate solutions will carry less weight than 4 complete ones.

You may use standard results, provided you carefully state them ***in full.***

Six problems carry **full credit.**

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(1) Let $\mathcal{C} = \mathrm{Mod}(R)$ be the category of R-modules, where R is a ring. Let $f : A \to M$ and $g : A \to N$ be two homomorphisms. Prove there exists an R-module, denoted $M \oplus_A N$, such that for any R-module P, and any pair of homomorphisms $u : M \to P$ and $v : N \to P$, such that $u \circ f = v \circ g$, there exists a unique homomorphism $M \oplus_A N \to P$. Also prove that such $M \oplus_A N$ is determined up to unique isomorphism.

(2) Let G be a group with $|G| = 2 \cdot 3 \cdot 5^3 = 750$. Prove that G cannot be simple.

(3) Let k be a field, and $R = k[x]/(x^N)$ for some $N > 0$. Denote by $\bar{x}$ the class of x in R. (Recall that in a commutative ring R the localization at a prime $\mathfrak{p}$ means $S^{-1}R$, where $S = R \setminus \mathfrak{p}$.)

Let M be an R-module such that $M_{(\bar{x})} = 0$. Prove that $M = 0$.

(4) Let $R = \mathbb{Z}[\sqrt{3}]$.

(i) Prove that $a \pm \sqrt{3}$, $a \in \mathbb{Z}$, is prime if and only if its norm $a^2 - 3$ is prime in $\mathbb{Z}$.

(ii) Let $f = x^n - r \in R[x]$. Assume $p = 13$ divides r, but p^2 does not. Prove that f is irreducible.

(5) Prove that the polynomial $f = x^3y^2 + x^2y^3 + x + y^3$ is irreducible in $k[x, y]$, where k a field, assumed to be of characteristic zero, for simplicity.

Is the ideal $I = (f)$ prime? Maximal?

(6) Let R be a commutative ring, and let I, J be two ideals. Fill the blanks in the following diagram in such a way that every row and column is short exact (justify your procedure):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I \cap J & \longrightarrow & J & \longrightarrow & ? & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & ? & \longrightarrow & R & \longrightarrow & R/I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & ? & \longrightarrow & ? & \longrightarrow & ? & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

(7) Let I be the ideal (p, x) in $R = \mathbb{Z}[x]$, where p is a prime.

(a) Prove that I is not a free R-module.

(b) Provide a free resolution.

(8) Let $G = \mathrm{GL}_3(\mathbb{F}_2)$, the group of invertible 3×3 matrices with entries in $\mathbb{F}_2$, the field with two elements. Let G act on $M_3(\mathbb{F}_2)$, the set of 3×3 matrices, by $A \mapsto gAg^{-1}$, where $g \in G$ and $A \in M_3(\mathbb{F}_2)$. Classify the orbits of this action.