12 War Over an Internal Prize. Conditions for Peace

In Lecture 11 we assumed that Nation i controls resources of value b_i for $i = 1, 2$, and that these two nations are in dispute over a prize b external to their dyad. We continue to make the first assumption, and we continue to interpret u and v as the efforts that Nations 1 and 2, respectively, expend on war, although we now measure effort in terms of its cost and we strengthen (11.13) to

$$b_1 \;>\; b_2 \tag{12.1}$$

(so that we can definitely regard Nation 1 as the rich nation and Nation 2 as the poor one). However, we now follow Beviá and Corchón (2010) in assuming that the prize over which Nations 1 and 2 are contemplating war is the collective resources of both nations, and hence is no longer external. Rather, if these two nations do go to war, then one of them ultimately acquires all of the resources that either one controls. So

$$b \;=\; b_1 + b_2. \tag{12.2}$$

As in Lecture 11, the efforts that Nations 1 and 2 expend on war, u and v, are both positive (unless there is no war). Moreover, because Nation i's war effort subtracts from its resources, which total b_i, we also require $s_i \leq b_i$. So

$$S_i \;=\; \big(0, b_i\big] \tag{12.3}$$

for $i = 1, 2$ and

$$D \;=\; \big(0, b_1\big] \times \big(0, b_2\big] \tag{12.4}$$

as anticipated by (11.18).

For the sake of simplicity, let us assume that the CSF is given by (11.7) with $\gamma = 1$, so that the probability of victory for either side increases with effort ratio in a way that is neither especially sensitive to it nor especially insensitive to it.[1] Then

$$p_1(u, v) \;=\; \frac{u}{u+v}, \qquad p_2(u, v) \;=\; 1 - p_1(u, v) \;=\; \frac{v}{u+v}. \tag{12.5}$$

A central assumption of Beviá and Corchón (2010) now comes into play. They assume that effort is of two kinds. For both players, a fraction $\kappa \in (0, 1)$ of war effort is not recoverable—it is truly spent—whereas fraction $1 - \kappa$ is recoverable by the winner as a kind a return on investment, although the winner reaps what the loser has sown (as well as what it has sown itself). So what remains of $\mathcal{P}_1$'s resources at war's end is not $b_i - s_i$, but rather $b_i - s_i + (1 - \kappa)s_i = b_i - \kappa s_i$. Because the winner acquires whatever remains of both players resources and the loser ends up with nothing, $\mathcal{P}_i$'s payoff is the random variable

$$F_i \;=\; \begin{cases} b_1 - \kappa s_1 + b_2 - \kappa s_2 & \text{if } \mathcal{P}_i \text{ wins} \\ 0 & \text{if } \mathcal{P}_i \text{ loses} \end{cases} \;=\; \begin{cases} b - \kappa(u+v) & \text{if } \mathcal{P}_i \text{ wins} \\ 0 & \text{if } \mathcal{P}_i \text{ loses} \end{cases} \tag{12.6}$$

[1] Needless to say, p_i may depend on other factors, but they are abstracted from the model.

by (12.2), implying

$$
\begin{aligned}
f_i(u,v) \;=\; \mathrm{E}[F_i] \;&=\; \{b - \kappa(u+v)\}\cdot p_i(u,v) + 0\cdot\{1 - p_i(u,v)\} \\
&=\; \{b - \kappa(u+v)\}\, p_i(u,v).
\end{aligned}
\tag{12.7}
$$

So, on using (12.5), the rewards are given by

$$
\begin{aligned}
f_1(u,v) \;&=\; \frac{bu}{u+v} - \kappa u \\
f_2(u,v) \;&=\; \frac{bv}{u+v} - \kappa v
\end{aligned}
\qquad \text{for } (u,v)\in D
\tag{12.8}
$$

where D is defined by (12.4). Note that (12.8) is the special case of (11.4) for which (11.5), (11.6) and (12.5) all hold. So κ—the fraction of war effort that is not recoverable—is also in effect its marginal cost.

We now obtain the reaction sets from calculations that parallel those in Lecture 11, because (12.8) is the special case of (11.17) in which $\lambda = 1$.[2] Let us first define

$$
\hat{u}(v) \;=\; \sqrt{\frac{bv}{\kappa}} - v, \qquad \hat{v}(u) \;=\; \sqrt{\frac{bu}{\kappa}} - u.
\tag{12.9}
$$

Then as in Lecture 11 (p. 73), we find that f_1 increases from 0 as $u \to 0$ to a maximum of $f_1(\hat{u}(v), v) = (\sqrt{b} - \sqrt{\kappa v})^2$ at $u = \hat{u}(v)$ before decreasing towards 0 as $u \to \frac{b}{\kappa} - v$. Moreover, as before, we find that $\hat{u}(v)$ increases from $\hat{u}(0) = 0$ to $\hat{u}(\frac{b}{4\kappa}) = \frac{b}{4\kappa}$ on $[0, \frac{b}{4\kappa}]$ before decreasing on $[\frac{b}{4\kappa}, \frac{b}{\kappa}]$ to $\hat{u}(\frac{b}{\kappa}) = 0$. So $\hat{u}(v)$ is certainly the best response to v when

$$
\frac{b}{4\kappa} \;<\; b_1,
\tag{12.10}
$$

which ensures $\hat{u}(v) \in S_1$ for all $v \in S_2$ to satisfy (12.3). Then $\mathcal{B}_1(v) = \hat{u}(v)$ for all $v \in S_2$. If

$$
b_1 \;<\; \frac{b}{4\kappa},
\tag{12.11}
$$

however, then $\hat{u}(v) > b_1$ for $v \in (v_-, v_+)$, where we define

$$
v_\pm \;=\; \tfrac{1}{2}\Big\{\tfrac{b}{\kappa} - 2b_1 \pm \sqrt{\tfrac{b}{\kappa}\big(\tfrac{b}{\kappa} - 4b_1\big)}\Big\}.
\tag{12.12}
$$

Because now $v \in (v_-, v_+)$ implies that $\partial f_1/\partial u > 0$ for all $u \in (0, b_1)$, the best reply to all such v becomes $u = b_1$. So

$$
\mathcal{B}_1(v) \;=\; \begin{cases} \hat{u}(v) & \text{if } 0 < v < v_- \\ b_1 & \text{if } v_- \le v \le b_2 \end{cases}
\tag{12.13}
$$

[2] Nevertheless, the calculations are not quite identical because b_1, b_2 depend on one another through (12.2), and especially because D in (12.4) is different from D in (11.15)—and we cannot just set $\kappa = 1$, because the rewards would then be wrong.

and part of R_1 must lie on the right-hand boundary of D whenever $v_- < b_2$, which straightforward algebra reduces to

$$b_2 \;>\; \kappa b \tag{12.14}$$

(in which case, $v_+ > b_2$, so that $v_+ \notin S_2$). An almost identical analysis shows that, for

$$\frac{b}{4\kappa} \;<\; b_2, \tag{12.15}$$

$\mathcal{B}_2(u) = \hat{v}(u)$ for all $u \in S_1$; whereas for

$$b_2 \;<\; \frac{b}{4\kappa} \tag{12.16}$$

(which $b_1 < \frac{b}{4\kappa}$ implies),

$$\mathcal{B}_2(u) \;=\; \begin{cases} \hat{v}(u) & \text{if } 0 < u < u_- \\ b_2 & \text{if } u_- \le u \le b_1 \end{cases} \tag{12.17}$$

where $\hat{v}(u)$ is defined by (12.9) with

$$u_\pm \;=\; \tfrac{1}{2}\Big\{\tfrac{b}{\kappa} - 2b_2 \pm \sqrt{\tfrac{b}{\kappa}\big(\tfrac{b}{\kappa} - 4b_2\big)}\Big\}; \tag{12.18}$$

and part of R_2 must lie on the upper boundary of D, because straightforward algebra shows that $u_- < b_2 < b_1$.

The implication of the above analysis is that three different types of Nash equilibrium arise according to whether (12.10) and (12.15) both hold, requiring $\kappa > \frac{1}{2}$ (Case I below); or (12.11), (12.14) and (12.16) all hold, requiring $\kappa < \frac{1}{2}$ (Case II below); or (12.16) holds either with (12.10), requiring $\kappa > \frac{1}{4}$, or with (12.11) when (12.14) is false (Case III below). Defining the dimensionless parameter

$$\beta \;=\; \frac{b_2}{b} \;=\; \frac{b_2}{b_1 + b_2} \tag{12.19}$$

will help in distinguishing between these cases.[3] Note that

$$\beta \;<\; \tfrac{1}{2} \tag{12.20}$$

by (12.1)–(12.2).

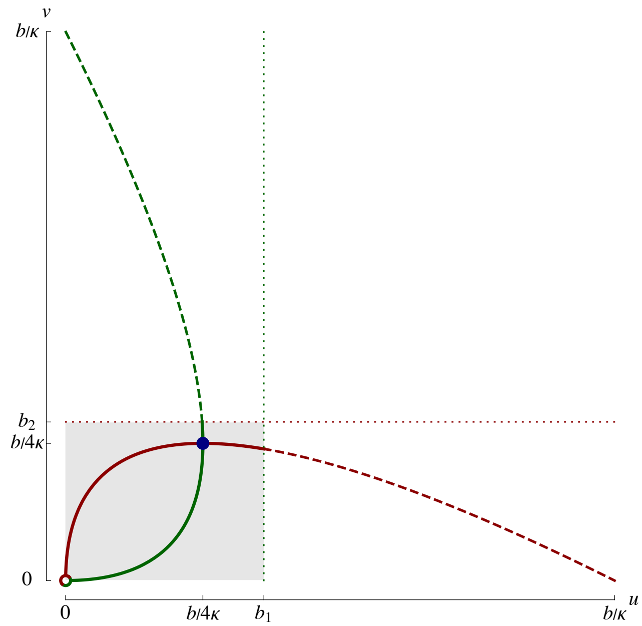
Case I: $\frac{b}{4\kappa} < b_2 < b_1$ or $1 - \beta > \beta > \frac{1}{4\kappa}$

Here $\mathcal{B}_1(v) = \hat{u}(v)$ for all $v \in S_2$ and $\mathcal{B}_2(u) = \hat{v}(u)$ for all $u \in S_1$, as illustrated by Figure 12.1. The Nash equilibrium occurs where R_1 and R_2 intersect at $(u^*, v^*) \in D$, that is, where $\hat{u}(v^*) = u^*$ and $\hat{v}(u^*) = v^*$ or

$$u^* \;=\; v^* \;=\; \frac{b}{4\kappa} \tag{12.21}$$

[3] In this regard, note that Case III means either $b_2 < \frac{b}{4\kappa} < b_1$ with $\kappa > \frac{1}{4}$ or $b_2 < \kappa b$ with $b_2 < b_1 < \frac{b}{4\kappa}$. The first possibility becomes $\beta < \min(\frac{1}{4\kappa}, 1 - \frac{1}{4\kappa})$; the second becomes $\beta < \kappa$, $\beta < 1 - \beta < \frac{1}{4\kappa}$ or $\beta < \min(\kappa, \frac{1}{4\kappa})$, which exceeds $\min(\frac{1}{4\kappa}, 1 - \frac{1}{4\kappa})$. So the first possibility is subsumed by the second.

Figure 12.1: The reaction sets R_1 (solid green), R_2 (solid red) and Nash equilibrium (blue dot) for $1-\beta > \beta > \frac{1}{4\kappa}$. The dashed curves are $u = \hat{u}(v)$ (green) and $v = \hat{v}(u)$. The decision set $D = S_1 \times S_2$ is shaded.



from (12.9). By (12.8), this equilibrium yields reward

$$w_i \;=\; f_i(u^*, v^*) \;=\; \tfrac{1}{4}b \tag{12.22}$$

to Player i—in the event of a war. In the absence of war, however, Player i's benefit would be b_i, which exceeds $\frac{1}{4}b$ for both $i = 1$ and $i = 2$ because $\frac{b}{4\kappa} < b_2 < b_1 \implies \frac{b}{4} < \kappa b_2 < b_2 < b_1$ for $\kappa \in (0,1)$. Thus

$$w_i \;<\; b_i \tag{12.23}$$

for $i = 1, 2$: both nations prefer peace to war in Case I.

We can interpret this result as saying that war does not pay because too small a fraction of war expenditure is recoverable after the war and the disparity in resources between the two nations is relatively small. With regard to the first point, note that Case I arises only for $\kappa > \frac{1}{2}$. With regard to the second point, note that for $\beta > \frac{1}{4\kappa}$ we have

$$1 \;<\; \frac{b_1}{b_2} \;=\; \frac{1}{\beta} - 1 \;<\; 4\kappa - 1, \tag{12.24}$$

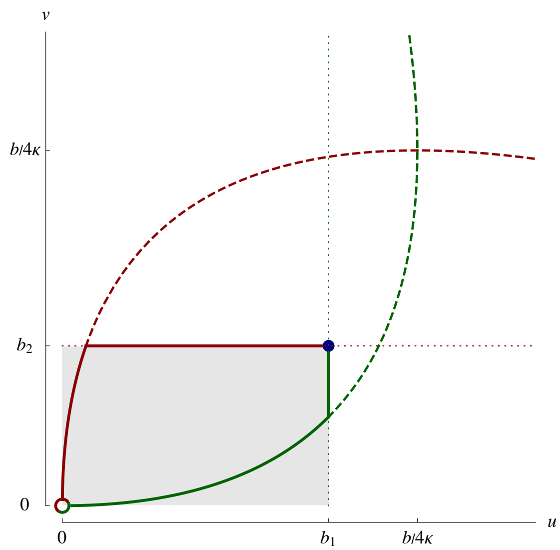
which cannot exceed 3 and is close to 1 if, as seems likely, κ is not much bigger than $\frac{1}{2}$.

Case II: $\kappa b < b_2 < b_1 < \frac{b}{4\kappa}$ or $\frac{1}{4\kappa} > 1 - \beta > \beta > \kappa$

Here we need both (12.13) and (12.17), as illustrated by Figure 12.2. We obtain

$$u^* \;=\; b_1, \qquad v^* \;=\; b_2 \tag{12.25}$$

Figure 12.2: The reaction sets R_1 (solid green), R_2 (solid red) and Nash equilibrium (blue dot) for $\frac{1}{4\kappa} > 1 - \beta > \beta > \kappa$. The dashed curves are $u = \hat{u}(v)$ (green) and $v = \hat{v}(u)$. The decision set $D = S_1 \times S_2$ is shaded.



at the Nash equilibrium (u^*, v^*), with

$$w_i \;=\; f_i(u^*, v^*) \;=\; (1 - \kappa)b_i \tag{12.26}$$

for $i = 1, 2$ by (12.8). Because $\kappa \in (0, 1)$, we have $w_i < b_i$ for $i = 1, 2$; and so, as in Case I, both players prefer peace to war in Case II (which arises only for $\kappa < \frac{1}{2}$). In this case, war does not pay because it is so destructive—it requires each side to commit all of its resources to it.

Case III: $b_2 < \kappa b$, $b_2 < \frac{b}{4\kappa}$ or $\min\left(\kappa, \frac{1}{4\kappa}\right) > \beta$

Here $\mathcal{B}_1(v) = \hat{u}(v)$ for all $v \in S_2$ but $\mathcal{B}_2(u) = \hat{v}(u)$ only for $u \le u_-$, with $\mathcal{B}_2(u) = b_2$ for $u_- \le u \le b_1$, as illustrated by Figure 12.3. Now R_1 and R_2 intersect where $v = v^* = b_2$ and $u = u^* = \hat{u}(v^*) = \hat{u}(b_2)$. Accordingly, by (12.9), we obtain

$$u^* \;=\; \sqrt{\frac{b_2 b}{\kappa}} - b_2, \qquad v^* \;=\; b_2 \tag{12.27}$$

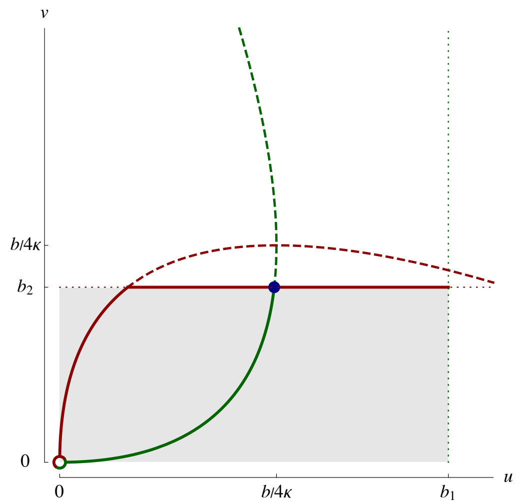
for the Nash equilibrium. The corresponding rewards are

$$\begin{aligned} w_1 \;&=\; f_1(u^*, v^*) \;=\; b - 2\sqrt{\kappa b_2 b} + \kappa b_2 \;=\; (\sqrt{b} - \sqrt{\kappa b_2})^2 \\ w_2 \;&=\; f_2(u^*, v^*) \;=\; \sqrt{\kappa b_2 b} - \kappa b_2 \;=\; \sqrt{\kappa b_2}(\sqrt{b} - \sqrt{\kappa b_2}) \end{aligned} \tag{12.28}$$

by (12.2) and (12.8). So in this case Nation 1 prefers peace to war if $b_1 \ge b - 2\sqrt{\kappa b_2 b} + \kappa b_2$, which readily reduces to

$$4\kappa b_1 - (1 - \kappa)^2 b_2 \;\ge\; 0. \tag{12.29}$$

Figure 12.3: The reaction sets R_1 (solid green), R_2 (solid red) and Nash equilibrium (blue dot) for $\min\left(\kappa, \frac{1}{4\kappa}\right) > \beta$. The dashed curves are $u = \hat{u}(v)$ (green) and $v = \hat{v}(u)$. The decision set $D = S_1 \times S_2$ is shaded.



But $4\kappa b_1 > b_2$ and $1 > (1-\kappa)^2$, so that (12.29) is always satisfied (with strict inequality) and (12.23) holds for $i = 1$: Nation 1 invariably prefers peace to war in Case III. By contrast, Nation 2 prefers peace to war in Case III only if $b_2 \geq \sqrt{\kappa b_2 b} - \kappa b_2$, which reduces to $(1+\kappa)^2 b_2 \geq \kappa b$ or

$$\beta \;\geq\; \frac{\kappa}{(1+\kappa)^2}. \tag{12.30}$$

If this inequality is reversed, that is, if

$$\beta \;<\; \frac{\kappa}{(1+\kappa)^2}, \tag{12.31}$$

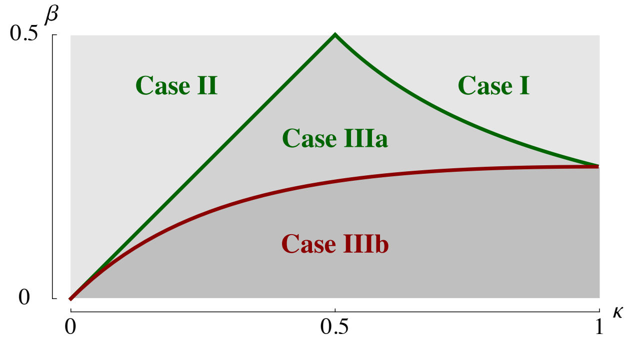
then Nation 2 prefers war to peace in Case III.

We can interpret (12.31) as saying that when β is sufficiently small, Nation 2 has an incentive to declare war because the resource disparity $\frac{1}{\beta} - 1 = b_1/b_2$ between richer and poorer is large enough for war to seem an attractive get-rich-quick option for the poorer nation. As noted by Beviá and Corchón (2010, p. 474), however, the result is counterintuitive, because $\frac{\kappa}{(1+\kappa)^2}$ increases with κ, making (12.31) is easier to satisfy when κ is larger than when κ is smaller—the more that cannot be recovered afterwards, the greater the incentive for war! The resolution of this paradox is, in essence, that u^* decreases with κ by (12.27), and so the probability that Nation 2 wins the contest and the corresponding expected payoff both increase with κ.[4]

These three cases are summarized by Figure 12.4. Above the red curve, where (12.30) is satisfied (Case IIIa), both sides prefer peace to war. So war should not be declared—even

[4]By (12.5) and (12.27), $p_2(u^*, v^*) = \sqrt{b_2\kappa/b}$, which is clearly increasing. By (12.28), we have $\partial w_2/\partial\kappa = \frac{1}{2}\sqrt{b_2}(\sqrt{b} - 2\sqrt{\kappa b_2}/\sqrt{\kappa}$, which is positive for $b_2 < \frac{b}{4\kappa}$.

Figure 12.4: War versus peace in the κ-β plane in the absence of a resource transfer.



without concessions. Below the red curve (Case IIIb), however, even though Nation 1 prefers peace to war, because Nation 2 prefers war to peace, it is rational for Player 2 to declare war—and so war will break out, because peace requires both sides to abstain. Thus the probability of war may be considerable in the absence of a peace agreement. For example, if the point (κ, β) is distributed over the rectangle $[0,1] \times [0, \frac{1}{2}]$ in Figure 12.4 uniformly, that is, with joint probability density function g defined by

$$g(\kappa, \beta) \;=\; 2 \tag{12.32}$$

so that

$$\int_0^1 \int_0^{\frac{1}{2}} g(\kappa, \beta)\, d\beta\, d\kappa \;=\; 1, \tag{12.33}$$

then the probability of war—the probability that (κ, β) lies below the red curve—is

$$P_w \;=\; \int_0^1 \int_0^{\frac{\kappa}{(1+\kappa)^2}} g(\kappa, \beta)\, d\beta\, d\kappa \;=\; 2\ln(2) - 1 \;\approx\; 0.3863. \tag{12.34}$$

We now ask whether Nation 1 can avert a war with Nation 2 in Case IIIb—that is, when (12.31) holds or

$$(\kappa, \beta) \;\in\; \text{IIIb}, \tag{12.35}$$

where IIIb denotes the region below the red curve in Figure 12.4—by making concessions in the form of a resource transfer. Let T denote the amount transferred by Nation 1 to Nation 2, so that Nation 1's resources fall from b_1 to $b_1 - T$ and Nation 2's rise from b_2 to $b_2 + T$. Thus, as a proportion of the prize b, Nation 2's resources rise from $b_2/b = \beta$ to $(b_2 + T)/b = \beta + \tau$, where we define

$$\tau \;=\; T/b. \tag{12.36}$$

In terms of Figure 12.4, the effect of this transfer is to raise the point (κ, β) vertically a distance τ to the point $(\kappa, \beta + \tau)$. Absent the transfer, war will break out when (12.35) holds because (12.30) implies $w_2 > b_2$, where w_i is defined by (12.28), and so

$$w_i \leq b_i \tag{12.37}$$

fails to be satisfied for both i. After the transfer, however, both nations will prefer peace to war if (and only if)

$$w_1 \;\leq\; b_1 - T, \qquad w_2 \;\leq\; b_2 + T. \tag{12.38}$$

On using (12.28) and (12.36), however, we can rewrite $w_2 \leq b_2 + T$ as $\sqrt{\kappa\beta} - \kappa\beta \leq \beta + \tau$ and $w_1 \leq b_1 - T$ as $\beta + \tau \leq 2\sqrt{\kappa\beta} - \kappa\beta$. Thus (12.38) reduces to

$$\sqrt{\kappa\beta} - \kappa\beta \;\leq\; \beta + \tau \;\leq\; 2\sqrt{\kappa\beta} - \kappa\beta. \tag{12.39}$$

Naturally, Nation 1 does not want to give away any more in resources than is necessary to prevent war. Hence the optimal transfer, $\tau = \tau^*$, is the one that raises the point (κ, β) by a vertical distance τ^* that is just enough for $(\kappa, \beta + \tau^*)$ to lie on the red curve, at which Nation 2 begins to prefer peace to war. So

$$\beta + \tau^* \;=\; \frac{\kappa}{(1+\kappa)^2} \tag{12.40}$$

by (12.30). Note that $\tau^* = \frac{\kappa}{(1+\kappa)^2} - \beta$ increases with κ, because Nation 2's incentive for war is greater when κ is greater.

However, this optimal transfer will actually prevent a war only if both nations prefer the reward that results from peace to the reward that results from war, that is, if (12.39) holds with $\tau = \tau^*$ or

$$\sqrt{\kappa\beta} - \kappa\beta \;\leq\; \frac{\kappa}{(1+\kappa)^2} \;\leq\; 2\sqrt{\kappa\beta} - \kappa\beta \tag{12.41}$$

by (12.40). As a consequence of (12.31) or equivalently (12.35), the first of these two inequalities is guaranteed to hold (strictly). The second inequality reduces to

$$\beta \;\geq\; \phi(\kappa), \tag{12.42}$$

where we define

$$\phi(\kappa) \;=\; \frac{2}{\kappa} - \frac{1}{(1+\kappa)^2} - \frac{2\sqrt{1+\kappa+\kappa^2}}{\kappa(1+\kappa)} \tag{12.43}$$

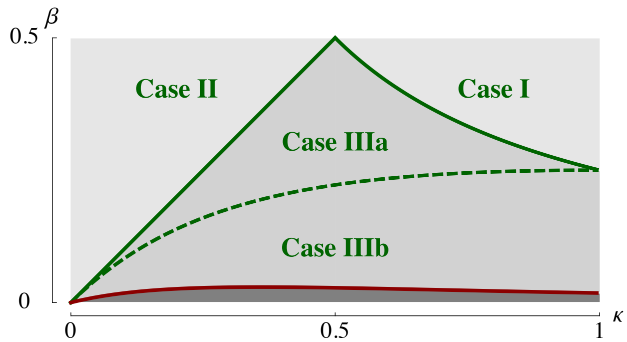
on $(0, \kappa)$. The curve with equation $\beta = \phi(\kappa)$ is the red curve in Figure 12.5. So it appears that (12.42) can be satisfied with relative ease. For example, if the point (κ, β) is uniformly distributed over the rectangle $[0, 1] \times [0, \frac{1}{2}]$ in Figure 12.5, then the probability of war is reduced from $P_w = 2\ln(2) - 1 \approx 0.3863$ in (12.34) to

$$p_w \;=\; \int_0^1 \int_0^{\phi(\kappa)} g(\kappa, \beta)\, d\beta\, d\kappa \;=\; 6\ln(3) - 8\ln(2) - 1 \;\approx\; 0.0465 \;\approx\; 0.1204 P_w, \tag{12.44}$$

that is, by factor of more than 8. So a transfer of resources cannot always prevent war, but makes it much less likely. Indeed, at least according to this model, war is impossible if

$$\beta \;>\; \beta_c, \tag{12.45}$$

Figure 12.5: War versus peace in the κ-β plane after a resource transfer.



where $\beta_c \approx 0.029$ is the maximum value of ϕ,[5] and (12.45) holds if and only if

$$\frac{b_1}{b_2} \;=\; \frac{1}{\beta} - 1 \;<\; \frac{1}{\beta_c} - 1 \;\approx\; 33.2. \tag{12.46}$$

Hence war is impossible unless the rich nation is more than 33 times richer than the poor nation.

In sum, if (κ, β) falls in any of the lighter shaded regions above the red curve in Figure 12.5, then either there is no incentive for war or war can be prevented by a transfer of resources. Either way there is peace. If (κ, β) falls in the darker region below the red curve, however, then peace is unsustainable, because Nation 2 has an incentive to declare war in the absence of a transfer, and the minimal transfer of resources that would prevent a war is too expensive for Nation 1. But it appears that no resource transfer can prevent a war only in circumstances where resource inequality could reasonably be regarded as extreme.

Beviá and Corchón extend their analysis to cases where either $\gamma \neq 1$ or $\lambda \neq 1$ in the CSF defined by (11.8) and discover, among other things, the possibility of a role reversal when $\lambda > 1$, in the sense that Nation 1 now has an incentive for war, which Nation 2 can prevent by transferring resources to Nation 1 (which therefore ends up even richer than it was to begin with). If any of you is sufficiently interested, perhaps their subsequent analysis could form the basis of a more in-depth investigation leading to an end-of-term presentation. For the rest of us, however, this is as far as we go.

[5] $\phi(\kappa)$ increases from 0 as $\kappa \to 0$ to $\phi(\kappa_{\max}) \approx 0.02923$ at $\kappa = \kappa_{\max}$ before decreasing again to $\frac{7}{4} - \sqrt{3} \approx 0.01795$ as $\kappa \to 1$, where $\phi'(\kappa_{\max}) = 0$ or

$$\kappa_{\max} \;=\; \frac{1}{3\sqrt{2}}\left\{\sqrt{14\sqrt{\frac{2}{3\sqrt[3]{6}-4}} - 3\sqrt[3]{6} - 8} - \sqrt{2} + \sqrt{3\sqrt[3]{6}-4}\right\} \;\approx\; 0.3577$$

(and $\phi''(\kappa_{\max}) \approx -0.18055 < 0$). Thus for all $\kappa \in (0, 1)$, we have $0 < \phi(\kappa) \leq \beta_c = \phi(\kappa_{\max}) \approx 0.02923$.